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An Improved Iterative Method Based on Cubic Spline Functionsfor Solving Nonlinear Equations

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Abstract: One new iterative method for solving nonlinear functions has been presented using a new quadrature rule based on cubic spline functions. This method has been found more efficient than Newton – Raphson method.

Keywords: Nonlinear equations, Newton's method, Improved iterative method, Cubic Spline functions, Numerical examples.

INTRODUCTION

For solving nonlinear equations, Newton's method is one of the most important methods in numerical analysis[1]. Recently, some methods have been proposed and analyzed for solving nonlinear equations [2-8]. These methods have been suggested by using quadrature formula, decomposition and Taylor's series [5, 9-12]. As we know, quadrature rules play an important and significant role in the evaluation of integrals. One of the most well – known iterative methods is Newton's classical method which has a quadratic convergence rate. Some authors have derived new iterative methods which are more efficient than that of Newton's [10, 11.13-15].

This paper has been organized as follows. Part 1 provides some preliminaries which are needed. Part 2 is devoted to suggest one new iterative method by using a new quadrature rule based on spline functions. These are implicit – type methods. To implement these methods, we use Newton's method as a predictor and then use one new method as a corrector. The resultant methods can be considered as two – step iterative methods. In Part 3, a comparison between this methods with that of Newton's methods. Several examples are given to illustrate the efficiencies and advantages of these methods.

DEFINITION AND NOTATION

Let $\alpha \in R$ and $x_N \in R, N = 0,1,2,3, \dots \dots \dots$ Then the sequence x_N is said to be convergence to α if $\lim_{N\to\infty} |x_N - \alpha| = 0$. If there exists a constant $c > 0$, an integer $N_0 \geq 0$ and $p \geq 0$ such that for all $N > N_0$

We have

$$|x_{N+1} - \alpha| \leq c|x_N - \alpha|^p$$

Then x_N is said to be converges to α with convergence order at least p. If $p = 2$, the convergence is to be quadratic or if $p = 3$ then it is cubic.

Notation: The notation e_n gives the formula for error $e_n = x_n - \alpha$, is the error in the n^{th} iteration.

The equation $e_{n+1} = ce_n^p + O(en^{p+1})$ is called the error equation. By substituting $e_n = x_n - \alpha$ for all n in any iterative method and simplifying. We obtain the error equation for that method. The value of p obtained is called the order of this method.

We consider the problem of numerical determine a real root α of non linear equation

$$f(x) = 0, \quad f: D \subset R \to R \tag{1.1}$$

The known numerical method for solving nonlinear equations is the Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2,3, \dots \dots \dots$$

Where x_0 is an initial approximation sufficiently near to α. The convergence order of the Newton's method is quadratic for simple roots[5].

Description of the Improved iterative method: For several reasons, we know that the cubic spline functions are popular spline functions .They are smooth functions with which to fit data, and when used for interpolation ,they do not have the oscillatory behavior that is the characteristic of high degree polynomial interpolation. We have derived a new quadrature rule which is based on spline interpolation[16]. In this section, we review it briefly and then present the iterative methods.

Let us suppose $\Delta = \{x_i | i = 0, 1, 2\}$ be a uniform partition of the interval $[a, b]$ by Knots $a = x_0 < \frac{a+b}{2} = x_1 \big(b = x_2$, and $h = x_1 - x_0 = x_2 - x_1$ and $y_i = f(x_i)\big)$; thenthe cubic spline function $S\Delta(x)$ which interpolates the values of the function f atthe knots $x_0, x_1, x_2 \in \Delta$ and satisfies $S''\Delta(a) = S''\Delta(b) = 0$ is readily characterizedby their moments, and these moments of interpolating cubic spline function can be calculated as the solution of a system of linear equations. We can obtain the following representation of the cubic spline function in terms of its moments[17].

$$S_\Delta(x) = \alpha_i + \beta_i(x - x_i) + \gamma_i(x - x_i)^2 + \delta_i(x - x_i)^3 \tag{2.1}$$

For $x \in [x_i, x_{i+1}]$, $i = 0,1$

Where

$$\alpha_i = y_i,\quad \beta_i = \frac{y_{i+1} - y_i}{h} - \frac{2M_i - M_{i+1}}{6}h,\qquad \gamma_i = \frac{M_{i+1} - M_i}{6h}$$

$$M_0 = M_2 = 0,\ \ M_1 = \frac{3}{2h^2}\left[f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Now from (2.1) we obtain

$$\int_a^b S_\Delta(x)dx = \int_a^{\frac{a+b}{2}} S_\Delta(x)dx + \int_{\frac{a+b}{2}}^b S_\Delta(x)dx$$

$$= \frac{h}{2}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{h^3}{24}[M_0 + 2M_1 + M_2]$$

$$= \frac{h}{8}\left[3f(a) + 10f\left(\frac{a+b}{2}\right) + 3f(b)\right]$$

And thus

$$\int_a^b f(t)dt \approx \frac{b-a}{16}\left[3f(a) + 10f\left(\frac{a+b}{2}\right) + 3f(b)\right] \tag{2.2}$$

We use the above quadrature rule to approximate integrals and use it to obtain an iterative method. In order to do this, let $\alpha \in D$ be a simple zero of sufficiently differentiable function $f: D \subset R \to R$ for an open interval D and x_0 is sufficiently close to α. To derive the iterative method, we consider the computation of the indefinite integral on an interval of integration arising from Newton's method.

$$f(x) = f(x_n) + \int_{x_n}^x f'(t)dt \tag{2.3}$$

And using the quadrature rule (2.2) to approximate the right integral of (2.3),

$$\int_{x_n}^x f'(t)dt = \frac{x-x_n}{16}\left[3f'(x_n) + 10f'\left(\frac{x+x_n}{2}\right) + 3f'(x)\right] \tag{2.4}$$

And looking for $f(x) = 0$. From (2.3) and (2.4) we obtain

$$x = x_n - \frac{16f(x_n)}{\left[3f'(x_n)+10f'\left(\frac{x_n+x}{2}\right)+3f'(x)\right]} \tag{2.5}$$

This fixed point formulation enables us to suggest the following implicit iterative method

$$x_{n+1} = x_n - \frac{16f(x_n)}{\left[3f'(x_n)+10f'\left(\frac{x_n+x_{n+1}}{2}\right)+3f'(x_{n+1})\right]} \tag{2.6}$$

Which requires the $(n+1)^{th}$ iterate x_{n+1} to calculate the $(n+1)^{th}$ iterate itself. To obtain the explicit form, we make use of the Newton's iterative step to compute $(n+1)^{th}$iterate x_{n+1} on the right hand side of (2.6), namely replacing $f\left(\frac{x_n+x_{n+1}}{2}\right)$ with$f\left(\frac{x_n+y_n}{2}\right)$, where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ is the Newton iterate and obtain the following explicit method.

Algorithm: For a givenx_0, compute the approximate solution x_{n+1} by the following iterative scheme

$$x_{n+1} = x_n - \frac{16f(x_n)}{\left[3f'(x_n) + 10f'\left(\frac{x_n+x_{n+1}}{2}\right) + 3f'(x_{n+1})\right]}, \qquad n = 0,1,2, \dots \dots \dots \dots$$

Where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

NUMERICAL EXPERIMENTS

In this section, we have employed the methods obtained in this paper to solve some nonlinear equations and compare them with the Newton's method. We have used the stopping criteria $|x_{n+1} - x_n| < \epsilon$ and $|f(x_{n+1})| < \epsilon$, where$\epsilon = 10 - 14$, for computer programs. All programs are written in Matlab.

The number of iterations (NI) and function evaluations (FE) is given so that the stopping criterion is satisfied.

We have used the following test functions with initial pointx_0 and display the approximate zeros $x*$ found up to the $14th$ decimal place.

Example 1:$f_1(x) = x^3 - x + 3, \quad x_0 = -1 \quad x^* = -1.67169988165716$

Example 2:$f_2(x) = x^3 + 4x^2 - 10, \quad x_0 = 1 \quad x^* = 1.36523001341410$

Example 3:$f_3(x) = -\cos x - x, \quad x_0 = 0 \quad x^* = -0.73908513321516$

Example 4: $f_4(x) = xe^x - \sin^2 x + 3\cos x + 5, \quad x_0 = -1 \quad x^* = -1.20764782713092$

Example 5: $f_5(x) = 3x - \cos x - 1, \quad x_0 = 0.5 \quad x^* = 0.60710164810312$

Example 6:$f_6(x) = xe^x - \cos x, x_0 = 0.5 \quad x^* = -0.51775736368246$

Example 7: $f_7(x) = x\log x - 4, \quad x_0 = 3 \quad x^* = 3.32732232259910$

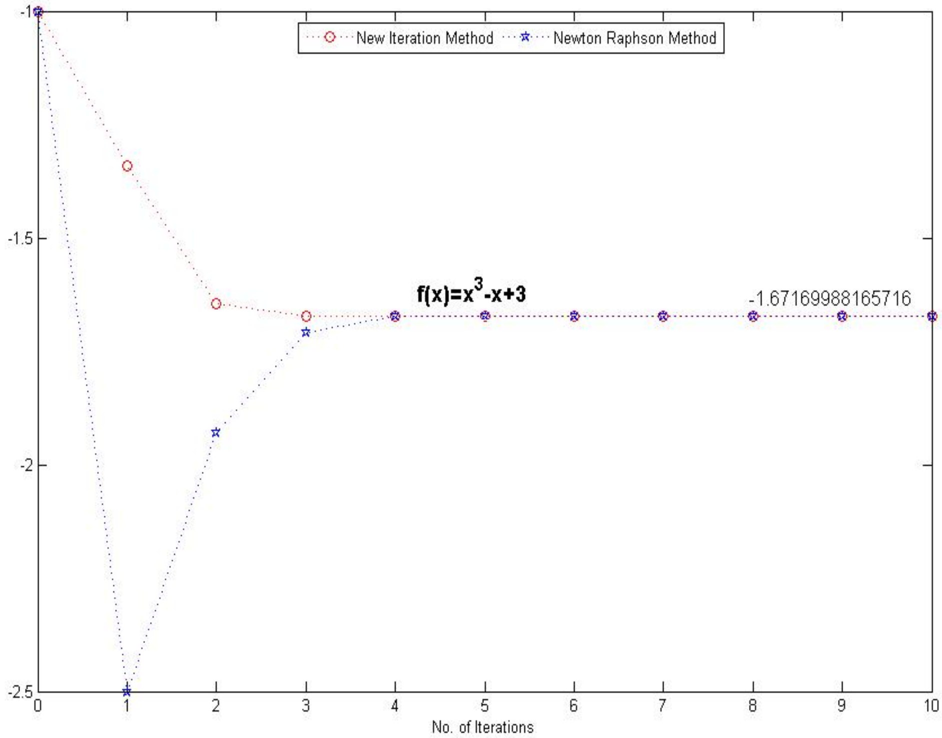
Example 8: $f_8(x) = 4x - e^x, \quad x_0 = 3 \quad x^* = 2.15329236411035$

Example 9:$f_9(x) = xtan\, x + 1 \quad x_0 = 2.5 \quad x^* = 2.79838604578389$

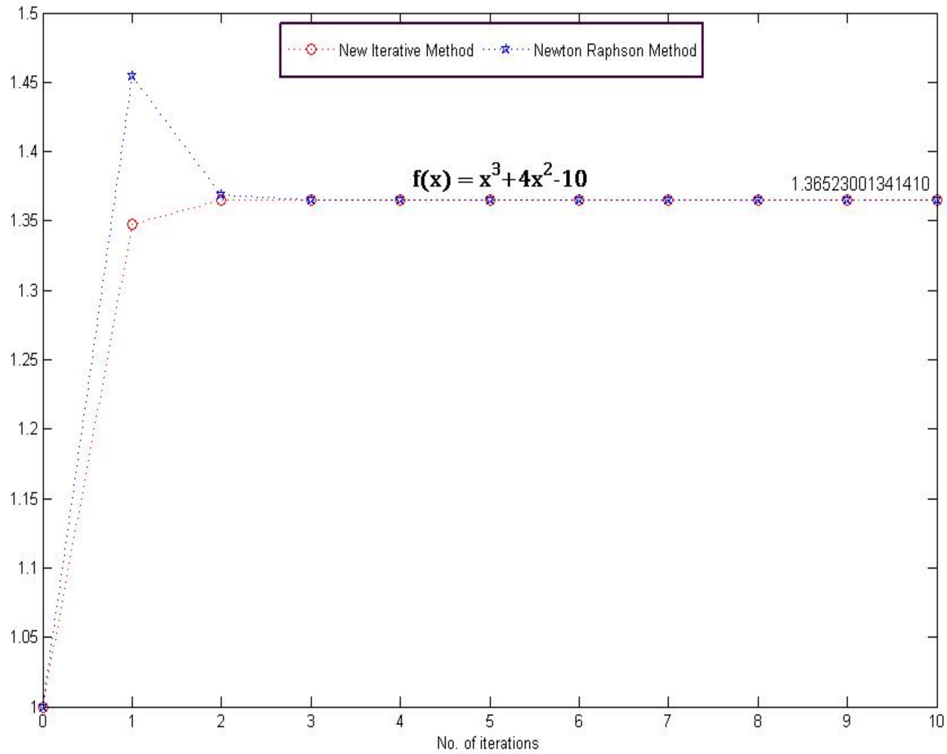
Example 10:$f_{10}(x) = x^3 - 3x^2 + 2.5 \quad x_0 = 2 \quad x^* = 2.64178352745293$

Comparison between the graphs of the New Iterative method, Newton's method

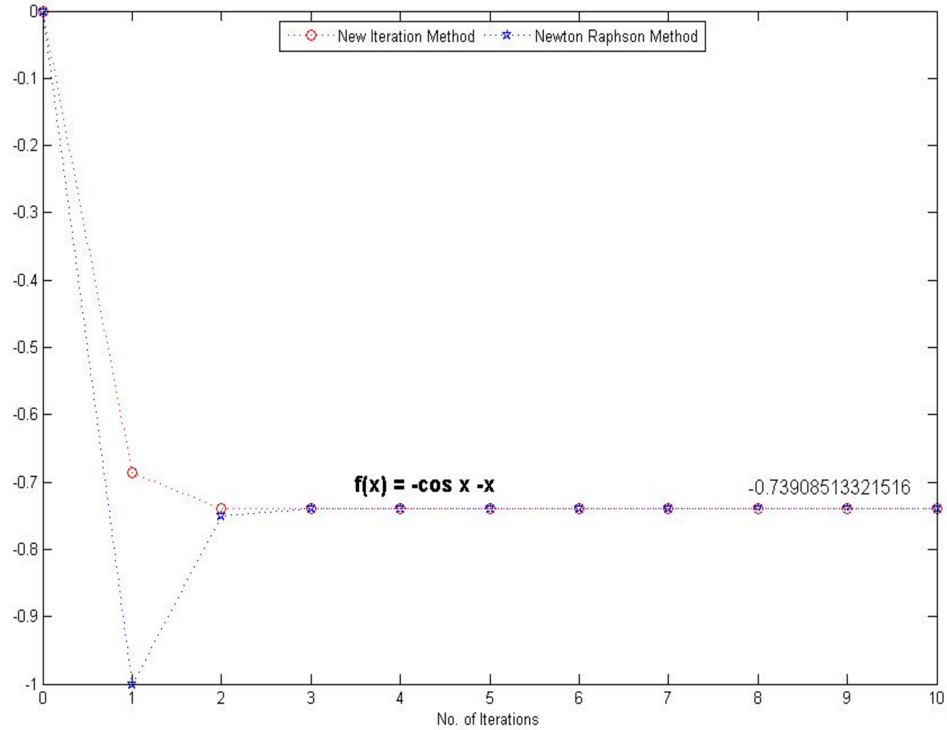
And number of Iteration.



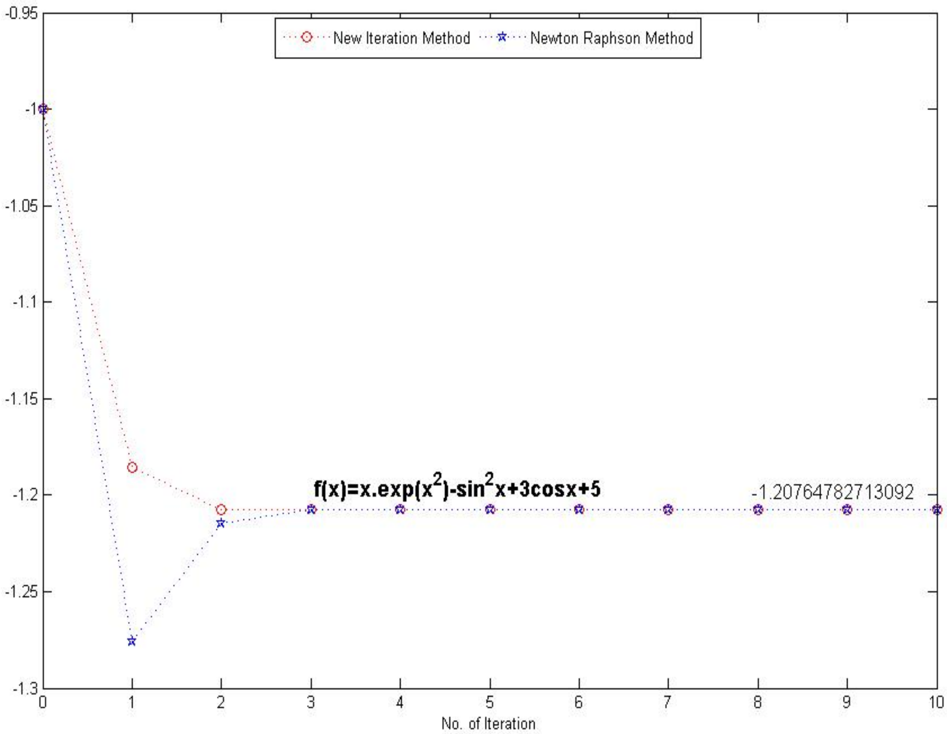
Graph for example: 1



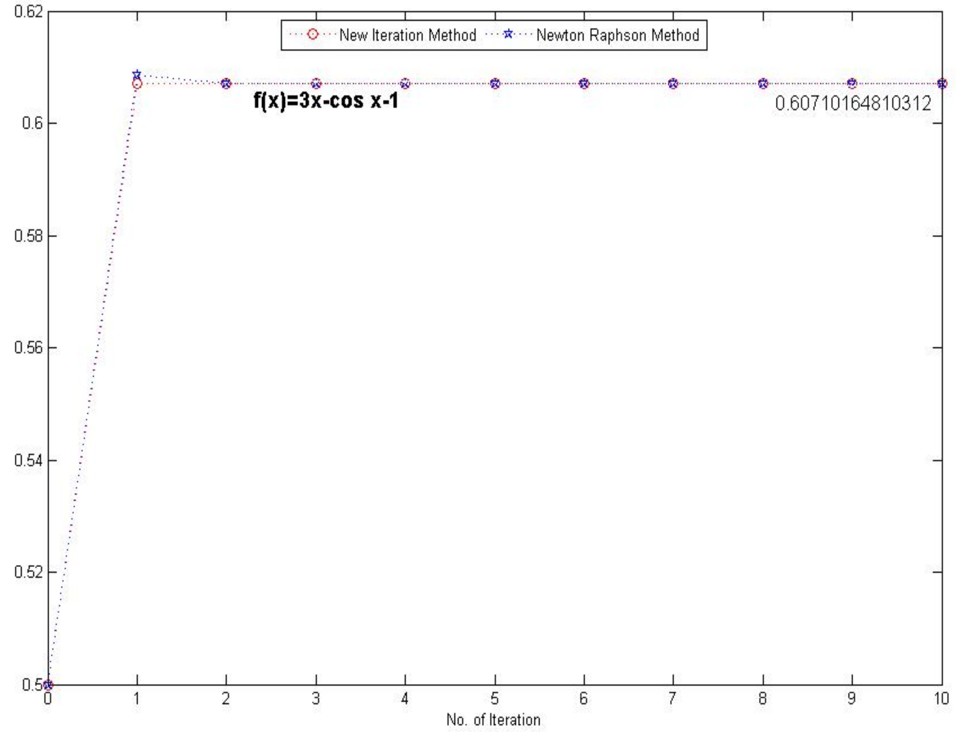
Graph for example: 2



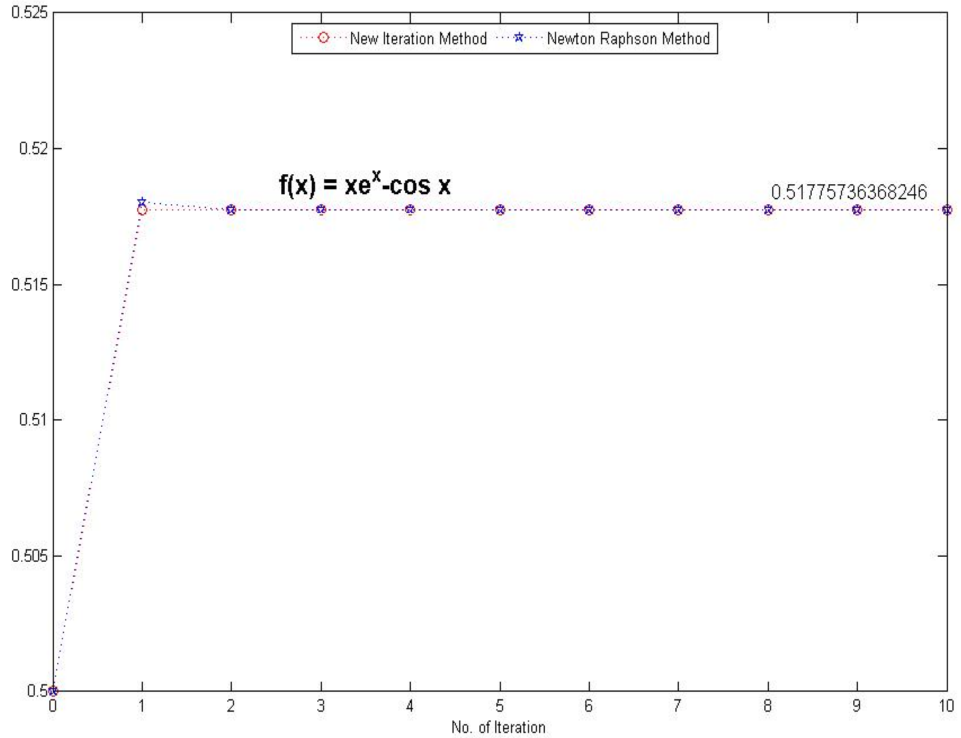
Graph for example: 3



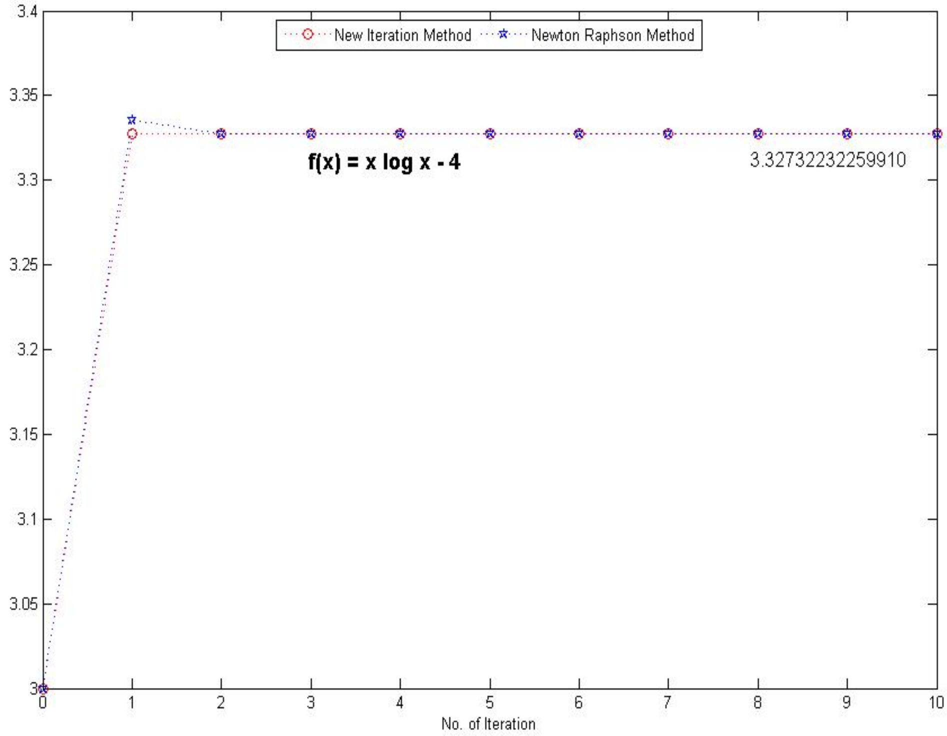
Graph for example: 4



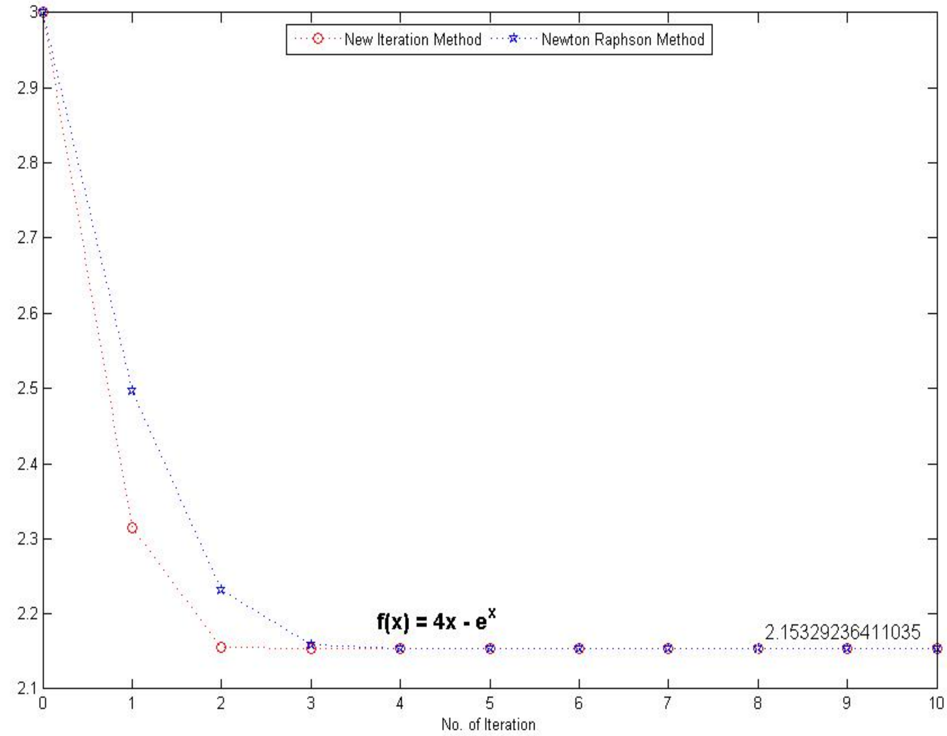
Graph for example: 5



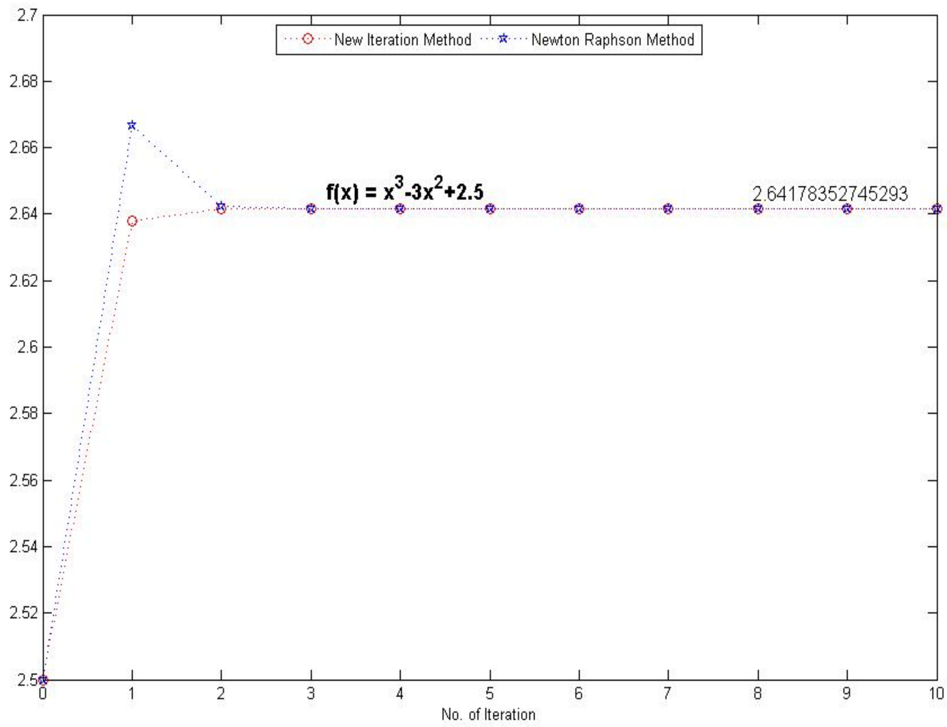
Graph for example: 6



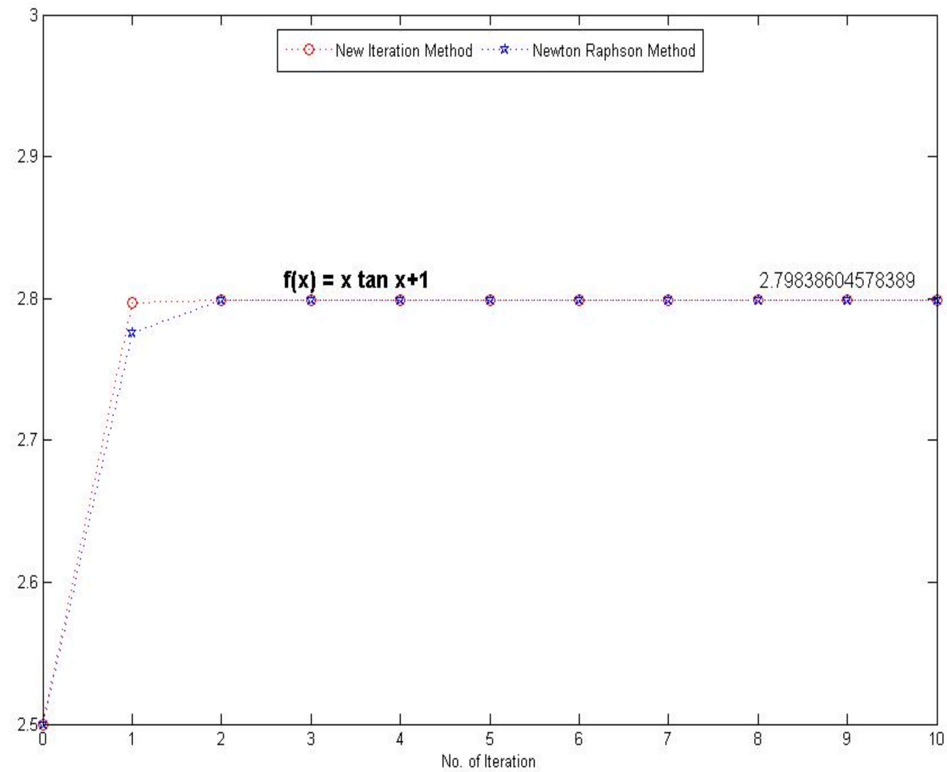
Graph for example: 7



Graph for example: 8



Graph for example: 9



Graph for example: 10

CONCLUSION

Analysis of efficiency from the graphs shows that this improved method is preferable to the well-known Newton's method. From numerical examples, it has been established that this methods is of great practical utility.

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