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An Admitting H-Projective Vector Field in Kaehlerian Spaces with Constant Scalar Curvature

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Abstract: Ishihara[1] has studied holomorphically projective changes and their groups in a almost complex manifold and also proved on holomorphic planes. Obata[2] has defined and studied Riemannian manifolds admitting a solution of a certain system of differential equations. Also, Negi and Gairola[3,4] have explain an admitting a conformal transformation group on Kaehlerian recurrent spaces. In this paper, I have defined and studied an admitting H-projective vector field in Kaehlerian spaces with constant scalar curvature and several theorems have been proved. Also, then find necessary and sufficient conditions for such a Kaehlerian space to be isometric to a complex projective space with Fubini-Study metric.

Keywords: Almost Hermition spaces, Kaehlerian spaces, H-projective vector field, associated covariant vector field, Fubini-Study metric.

MSC 2010: 53C15, 53C50, 53C55, 53C80, 53B35.

INTRODUCTION

Let *M* be a connected Kaehlerian spaces of complex dimension **n** covered by a system of real coordinate neighborhood (U ; x^h), where, here and in the sequel the indices h,i,j,k,……,run over the range {1,2,…..,2n}, and let g_{ji}, $F^h{}_i$ $\{_j{}^h{}_i\}$,∇_i , $K^h{}_{kji}$, K_{ji} , and K be the Hermition metric tensor, the complex structure tensor, the Christoffel symbols formed with g_{ji} , the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$, the curvature tensor, the Ricci tensor and scalar curvature of M respectively.

A vector field v^h is called a holomorphically projective (so called H-projective) vector field Ishihara[5], Yano[6], if it satisfies

(1.1) $\mathcal{L}_v\{{}_j{}^h{}_i\} = \nabla_j \nabla_i v^h + v^k K_{kji}^h = \rho_j \delta_i^h + \rho_i \delta_j^h - \rho_s F_J^s F_i^h - \rho_s F_i^s F_J^h$

For a certain covariant vector field ρ_j on M called the associated covariant vector field of v^h, where $\mathcal{L}_v$ denotes the operator of Lie- derivative with respect to v^h. In particular, if ρ_j is the zero-vector field, then v^h is called an affine vector field. Also, if we refer in the sequel to an H-projective vector field v^h, then we always mean by ρ_j the associated covariant vector field taking in (1.1).

Now, Let M be a Kaehlerian space of complex dimension n. The complex structure tensor F_i^h and the Hermitian metric tensor structure g_{ji} satisfy

(1.2) $F_i^h F_j^i = -\delta_j^h, \quad \nabla_j F_i^h = 0, \nabla_j F_{ih} = 0,$

(1.3) $F_j^s g_{si} = -F_i^s g_{js}$

(1.4) $g_{ji} = -F_j^t F_i^s g_{ts}$

We have Yano (1965), for the curvature tensor K_{kji}^h,

(1.5) $F_s^h K_{kji}^s = F_i^s K_{kjs}^h$

Or equivalently

(1.6) $K_{kji}^h = -F_i^t F_s^h K_{kjt}^s$

(1.7) $F_h^s K_{kjis} = -F_i^s K_{kjsh}$

(1.8) $K_{kjih} = F_i^t F_h^s K_{kjts}$, where $K_{kjih} = K_{kji}^t g_{th}$

Using (1.5) and the identity $K_{kji}^h + K_{ikj}^h + K_{jik}^h = 0$,

We obtain

$$F_s^h K_i^s = g^{ut} F_s^h K_{iut}^s = F^{ts} K_{its}^h = \frac{1}{2} F^{ts} (K_{its}^h - K_{ist}^h) = -\frac{1}{2} F^{ts} K_{tsi}^h$$

Where g^{ji} are contravariant components of g_{ji} *and* $F^{ts} = g^{ti} F_i^s$, that is,

(1.9) $F_s^h K_i^s = -\frac{1}{2} F^{kj} K_{kji}^h$,

From which it follows that

(1.10) $F_i^s K_{hs} = -\frac{1}{2} F^{kj} K_{kjih}$.

For the Ricci tensor K_{ji}, (1.9) we have

(1.11) $F_i^s K_s^h = F_s^h K_i^s$

or equivalently

(1.12) $K_i^h = -F_i^t F_s^h K_t^s$

Similarly, from (1.10) we have

(1.13) $F_j^s K_{si} = -F_i^s K_{js}$

or equivalently

(1.14) $K_{ji} = F_j^t F_i^s K_{ts}$

A vector field u^h on M is said to be contravariant analytic if

(1.15) $F_j^s \nabla_s u_i = -F_i^s \nabla_j u_s$

or equivalently

(1.16) $\nabla_j u_i = F_j^t F_i^s \nabla_t u_s$,

Where $u_i = g_{ih} u^h$, Since

$$\mathcal{L}_v F_i^h = -F_i^s \nabla_s u^h + F_s^h \nabla_i u^s = -(F_i^t \nabla_t u_s + F_s^t \nabla_i u_t)\, g^{sh},$$

a vector field u^h on M is contravariant analytic if and only if

(1.17) $\mathcal{L}_v F_i^h = 0$

Holds, where $\mathcal{L}_v$ denotes the operator of Lie derivation with respect to u^h.

It is known Yano (1965) that if M is compact, then a necessary and sufficient condition for a vector field u^h and M to be contravariant analytic is that

(1.18) $\nabla^j \nabla_j u^h + K_i^h u^i = 0$

Holds, where $\nabla^j = g^{ji} \nabla_i$.

For an H-projective vector field v^h on M defined by (1.1), we have

(1.19) $\nabla_j \nabla_s v^s = 2(\mathrm{n}+1)\, \rho_j$,

(1.20) $\nabla^j \nabla_j v^h + K_i^h v^i = 0$

(1.19) shows that the associated covariant vector field ρ_j is gradient. Putting

(1.21) $\rho = \frac{1}{2\,(n+1)} \nabla_s\, v^s$

We have

(1.22) $\rho_j = \nabla_j\, \rho$.

If an H- projective vector field v^h on M is contravariant analytic, then substituting (1.1) in the well-known formula Yano (1965, 1970)

$$\mathcal{L}_v\, K_{kji}^h = \nabla_k\, \mathcal{L}_v \left\{ {}_j{}^h{}_i \right\} - \nabla_j\, \mathcal{L}_v \left\{ {}_k{}^h{}_i \right\}$$

And using a straightforward computation we find

(1.23) $\mathcal{L}_v\, K_{kji}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i + (F_k^h \nabla_j \rho_s - F_j^h \nabla_k \rho_s) F_i^s$

$$+ (F_k^s \nabla_j \rho_s - F_j^s \nabla_k \rho_s)\, F_i^h,$$

From which by contracting with respect to **h** and **k** , we obtain

(1.24) $\mathcal{L}_v K_{ji} = -\ 2n\nabla_j\rho_i - 2\ F_j^{\ t}\ F_i^{\ s}\ \nabla_t\ \rho_s$.

Obata[2] has defined and studied let M be a complete connected and simply connected Kaehlerian manifold. In order for M to admit a nontrivial solution φ of a system of Partial differential equations

$$\nabla_j\nabla_i\varphi_h + \frac{c}{4}\left(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji}F_h^s\varphi_s - F_{jh}F_i^{\ s}\varphi_s\right) = 0$$

With a constant c>0, where $\varphi_h = \nabla_h\,\varphi$ and $F_{ji} = F_j^{\ t}\ g_{ti,}$ it is necessary and sufficient that M be isometric to a complex projective space CP[n] with Fubini-Study metric and of constant holomorphic sectional curvature **c**.

2. KAEHLERIAN SPACES WITH CONSTANT SCALAR CURVATURE

A Kaehlerian space M has the constant holomorphic sectional curvature k if and only if

(2.1) $K_{kji}^h = \frac{k}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k^{\ h} F_{ji} - F_j^{\ h} F_{ki} - 2F_{kj}F_i^{\ h})$

We define tensor fields G_{ji} and Z_{kji}^h on M by

(2.2) $G_{ji}\ = K_{ji} - \frac{k}{2n} g_{ji}$

(2.3) $Z_{kji}^h\ = K_{kji}^h - \frac{k}{4n(n+1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k^{\ h} F_{ji} - F_j^{\ h} F_{ki} - 2F_{kj}F_i^{\ h})$

Respectively. We then easily see that the tensor fields G_{ji} and Z_{kji}^h satisfy

(2.4) $G_{ji} = G_{ij},\ \ G_{ji}\ g^{ji} = 0,\ \ \ Z_{tji}^t = G_{ji},$

(2.5) $Z_{kjih}\ = -Z_{jkih},\ \ \ Z_{kjih} = \ Z_{ihkj,}$

(2.6) $Z_{kji}^h\ +\ Z_{ikj}^h\ +\ Z_{jik}^h = 0,$

Where $Z_{kjih}\ =\ Z_{kji}^t\,g_{th}$. If $G_{ji} = 0$, then M is a Kaehler-Einstein space and K is a constant provided n > 1, if Z_{kji}^h = 0, then M is of constant holomorphic sectional curvature $K/_{n(n+1)}$ provided n > 1.

Now, we have the following

Theorem (2.1): If an H-projective vector field v^h on a Kaehlerian space M of complex dimension n > 1 is contravariant analytic, then the associated vector field ρ^h is also contravariant analytic, and

(2.7) $\mathcal{L}_v K_{ji} = -2(n+1)\,\nabla_j\,\rho_i,$

Where $\rho^h = \rho_i\ g^{ih}$.

Proof: Applying the operator ρ_v of Lie derivation with respect to v^h to both sides of (1.14) and using $\mathcal{L}_v\ F_i^h = 0$, we have

$$\mathcal{L}_v\,K_{ji}\ =\ F_j^{\ t}\ F_i^{\ s}\ \mathcal{L}_v\,K_{ts},$$

From which together with (1.24) we see that ρ^h is contravariant analytic and (2.7) holds. Then, we have

Theorem (2.2): If a Kaehlerian space M is compact, then an H-projective vector field v^h on M is contravariant analytic, and consequently $\mathcal{L}_v F_i^h = 0$, Moreover, if n > 1, then the associated vector field ρ^h is contravariant analytic.

Theorem (2.3): For a contravariant analytic H-projective vector field v^h on a Kaehlerian space M with constant scalar K of complex dimension n > 1, we have

(2.8) $$\mathcal{L}_v G_{ji} = -\nabla_j w_i - \nabla_i w_j,$$

Where we have put

(2.9) $$w^h = (n+1)\rho^h + \frac{K}{2n} v^h,$$

And $w_i = g_{ih} w^h$

Proof: This follows from (2.2), (2.7) and the fact that ρ_j is gradient, that is $\rho_j = \nabla_j \rho$.

Theorem (2.4): For an H-projective vector field v^h on a compact Kaehlerian space M, we have

(2.10) $$\int_M \rho f \, dV = -\frac{1}{2(n+1)} \int_M \mathcal{L}_v f \, dV$$

For any real function f on M, where dV denotes the volume element of M, and ρ is the function defined by (1.21).

Proof: This follows from (1.21) and

$$0 = \int_M \nabla_i (f v^i) \, dV = \int_M f \nabla_i v^i dV + \int_M v^i \nabla_i f \, dV.$$

Theorem (2.5): In a compact Kaehlerian space M, we have

(2.11) $$\int_M \mathcal{L}_{Df} h \, dV = \int_M \mathcal{L}_{Dh} f \, dV = \int_M (\nabla_i f)(\nabla^i h) dV = f \Delta h \, dV = -\int_M h \Delta f dV$$

For any real functions f and h on M, where $\mathcal{L}_{Df}$ denotes the operator of Lie derivation with respect to the vector to the vector field $\nabla^i f$,and $\Delta = g^{ji} \nabla_j \nabla_i$.

Proof: This follows from

$$0 = \int_M \nabla_i (f \nabla^i h) dV = \int_M (\nabla_i f)(\nabla^i h) dV = \int_M f \Delta h \, dV,$$

$$0 = \int_M \nabla_i (h \nabla^i f) dV = \int_M (\nabla_i h)(\nabla^i f) dV = \int_M h \Delta f dV.$$

Theorem (2.6): If, in a compact Kaehlerian space M, a nonconstant function φ satisfies

(2.12) $$\nabla_j \nabla_i \varphi_h + \frac{c}{2}\left(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji} F_h^s \varphi_s - F_{jh} F_i^s \varphi_s\right) = 0.$$

Where $\varphi_h = \nabla_h \varphi$, c being a real constant, then the constant c is necessarily positive.

Proof: Transvecting (2.12) with g^{ih}, we have

$$\nabla_j \Delta\varphi + (n+1)c\, \varphi_j = 0$$

From which and Theorem (2.5) it follows that

$$c\int_M \varphi_j\varphi^j dV = -\frac{1}{n+1}\int_M (\nabla_j\Delta\varphi)\varphi^j dV = \frac{1}{n+1}\int_M (\Delta\varphi)^2\ dV$$

Where$\varphi^j = g^{ij}\varphi_i$. Since φ is a nonconstant function, two inequalities

$$\int_M \varphi_j\varphi^j dV > 0, \quad \int_M (\Delta\varphi)^2\ dV > 0.$$

Hold, and consequently C is necessarily positive.

Theorem (2.7): If a Kaehlerian space M with constant scalar curvature K admits an H-projective vector field v^h, and the vector field w^h

Defined by (2.9) is a Killing vector field, then the associated covariant vector field ρ_j satisfies

(2.13) $\nabla_j\nabla_i\rho_h + \frac{k}{4n(n+1)}\left(2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h^s\rho_s - F_{jh}F_i^{\ s}\rho_s\right) = 0$

Moreover, if M is complete and simply connected, K is positive and v^h is non-affine, then M is isometric to a complex projective space CP^n with Fubini-Study metric of constant holomorphic sectional curvature $k/n(n+1)$.

Proof: By using (1.1), we have

(2.14) $\nabla_j(\nabla_i v_h + \nabla_h v_i) = 2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h^s\rho_s - F_{jh}F_i^{\ s}\rho_s$

If w^h is a Killing vector field, then

$$\nabla_i w_h + \nabla_h w_i = 0,$$

Holds, and consequently

$$2(n+1)\nabla_i\rho_h + \frac{k}{2n}(\nabla_i v_h + \nabla_h v_i) = 0,$$

Which together with (2.14) implies (2.13). The second part of the lemma proved, if M is compact, then we can remove the positiveness of the scalar curvature K.

Theorem (2.8): For an H-projective vector field v^h on M is a compact Kaehlerian space of complex dimension $n > 1$ with constant scalar curvature K, we have

(2.15) $\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV = 2\int_M (\nabla_t w^t)^2\ dV$

Proof: By using a well-known integral formula YanoA[6,7] on a compact orientable Riemannian space, we have

$$\int_M (\nabla^j\nabla_j w^h + K_i^h w^i)\, w_h dV - \int_M (\nabla_t w^t)^2\ dV$$
$$+\frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV = 0$$

On the other hand, by Theorem (2.2) the associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^j\nabla_j\rho^h + K_i^h\rho^i = 0.$$

Consequently (2.15) follows immediately from (1.20) and the above relations since K is a constant.

Theorem (2.9): For an H-projective vector field v^h on M, we have

(2.16) $\int_M G_{ji}\rho^j w^i dV$

$$= \frac{1}{4(n+1)} \int_M (\nabla_j w_i + \nabla_i w_j)\,(\nabla^j w^i + \nabla^i w^j)\, dV.$$

Proof: From Theorem (2.2), the associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^j \nabla_j \rho^i + K_j^i \rho^j = 0,$$

From which and the equality

$$\nabla_i \nabla_t \rho^t = \nabla^t \nabla_t \rho_i - K_{ji}\rho^j$$

We fined

$$\nabla_i \nabla_t \rho^t = -2\, K_{ji}\, \rho^j.$$

Using the above equation (1.19),(2.2),(2.9) and Theorem(2.8), we have

$$\int_M G_{ji}\rho^j w^i dV = -\frac{1}{2} \int_M (\nabla_i \nabla_t \rho^t) w^i dV$$

$$- \frac{k}{4n(n+1)} \int_M (\nabla_i \nabla_t v^t) w^i\, dV$$

$$= - \frac{1}{2(n+1)} \int_M (\nabla_i \nabla_t w^t) w^i\, dV$$

$$= \frac{1}{2(n+1)} \int_M (\nabla_t w^t)^2\, dV$$

$$= \frac{1}{4(n+1)} \int_M (\nabla_j w_i + \nabla_i w_j)\,(\nabla^j w^i + \nabla^i w^j)\, dV.$$

3. COMPLEX PROJECTIVE SPACE

We have the following

Theorem (3.1): A complete simply connected Kaehlerian space M of complex dimension $n > 1$ with positive constant scalar curvature K admits a nonaffine and contravariant analytic H-projective vector field v^h such that

(3.1) $\mathcal{L}_v G_{ji} = 0,$

If and only if M *is* a isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional curvature$k/n(n+1)$.

Proof: This follows from Theorem (2.3) and (2.7).

Theorem (3.2): A complete simply connected Kaehlerian space M of complex dimension $n > 1$ with positive constant scalar curvature K admits a nonaffine and contravariant analytic H-projective vector field v^h such that

(3.2) $\mathcal{L}_v Z_{kji}^h = 0,$

If and only if M *is* a isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional curvature $k/n(n+1)$.

Proof: If (3.2) holds, then we have $\nabla_t w^t = 0$ and hence w^h

Is a Killing vector field. Consequently the theorem follows from theorem (2.7)

Theorem (3.3): A compact Kaehlerian space M of complex dimension n > 1 with constant scalar curvature K admits an H-projective vector field v^h. We have

(3.3) $$\int_M G_{ji}\rho^j w^i dV \geq 0,$$

Where w^i is defined by (2.9). Assume moreover that M is simply connected and v^h is nonaffine, then the equality in (3.3) holds if and only if M is isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional curvature $k/n(n+1)$.

Proof: This follows from Theorem (2.7) and (2.9).

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