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The Pseudo Inverse of a Partitioned Matrix

$$M = \begin{bmatrix} A_{11} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}$$

G.Ramesh[1] and B.K.N. Muthugobal[2]

[1]Ramanujan Research Centre,Department of Mathematics,
Government Arts College (Autonomous),Kumbakonam .

[2]Bharathidasan University constituent Arts and Science College, Nannilam.
Tamil Nadu – India.

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INTRODUCTION

If A_{11} is an mxn matrix over the complex field, then the Pseduo-inverse of A_{11}, denoted $A_{11}^{\dagger}$, is an nxm matrix such that

$$A_{11}A_{11}^{\dagger}A_{11} = A_{11} \qquad \ldots (1.1)$$

$$A_{11}^{\dagger}A_{11}A_{11}^{\dagger} = A_{11}^{\dagger} \qquad \ldots (1.2)$$

$$\left(A_{11}A_{11}^{\dagger}\right)^{*} = A_{11}A_{11}^{\dagger} \qquad \ldots (1.3)$$

$$\left(A_{11}^{\dagger}A_{11}\right)^{*} = A_{11}^{\dagger}A_{11} \qquad \ldots (1.4)$$

Any matrix which satisfies equation (1.1) is called an 1-inverse of A_{11}. A generalized inverse of A_{11} will indicate a matrix X satisfying some of the conditions (1.1) through (1.4).

If $M = \begin{bmatrix} A_{11} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}$ and M is invertible, then M^{-1} is lower block triangular. It is natural then to ask the following question.

For an mxn partitioned matrix

$$M = \begin{bmatrix} A_{11} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix} \qquad \ldots(1.5)$$

when is the Pseduo-inverse also lower block triangular? C. Meyer has given necessary and sufficient conditions for this question in [2].

We first give a formula for computing $M^{\dagger}$, and then we obtain Meyer's result as a corollary to this general expansion. We also examine some other cases which occur rather naturally.

In [3], Meyer considers square matrices which are upper triangular and he determines conditions for a generalized inverse to be upper triangular. Moreover, he gives explicit formulas for determining these inverses in some special cases.

Throughout our paper, we shall restrict our attention (except for a fleeting reference to 1-inverses) to the Pseduo-inverse. We shall use the following well-known facts in our work [e.g., see 4].

$$A_{11}^{\dagger} = A_{11}^{*}\left(A_{11}A_{11}^{*}\right)^{\dagger} = \left(A_{11}^{*}A_{11}\right)^{\dagger} A_{11}^{*} \qquad \ldots(1.6)$$

$$\left(A_{11}A_{11}^{*}\right)^{\dagger} = \left(A_{11}^{\dagger}\right)^{*} A_{11}^{\dagger} \qquad \ldots(1.7)$$

If $N(A_{11})$ denotes the null column space of A_{11}, then $N\left(A_{11}\right) \subset N\left(B\right)$ if and only if $B = BA_{11}^{\dagger}A_{11}$.

$$\ldots(1.8)$$

Lemmas: In order to prove our Theorem, we need the following lemmas.

Lemma 2.1

For M partitioned as in (1.5), we have $M^{\dagger} = \begin{bmatrix} A_{11}^{\dagger} & \begin{bmatrix} A_{12} \\ A_{13} \end{bmatrix}^{*} L^{\dagger} \\ 0 & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} L^{\dagger} \end{bmatrix}$

$L = \begin{bmatrix} A_{12} \\ A_{13} \end{bmatrix}\begin{bmatrix} A_{12} \\ A_{13} \end{bmatrix}^{*} + \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*}$, if and only if $\left[A_{11}\right]\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = 0$

Proof:

Assume $M^{\dagger} = \begin{bmatrix} {A_{11}}^{\dagger} & \begin{bmatrix} A_{12} \\ A_{13} \end{bmatrix}^{*} L^{\dagger} \\ 0 & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} L^{\dagger} \end{bmatrix}$ then by (1.1), we obtain

$$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} {A_{11}}^{\dagger} A_{11} + LL^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \quad \text{…(2.2)}$$

By the definition of L, we have N [L]⊆ N$\begin{bmatrix} A_{12} \\ A_{13} \end{bmatrix}^{*}$, so $LL^{+} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$ (1.8)

Then (2.2) implies $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} {A_{11}}^{\dagger} A_{11} = 0,$ and hence $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} {A_{11}}^{\dagger} = 0.$ But $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} {A_{11}}^{\dagger} = 0$ is equivalent to $A_{11} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} = 0,$ so the necessity is Complete. For the sufficiency, we will use relation (1.6) we have $\mathrm{M}^{\dagger} = M^{*} \left(M \quad M^{*} \right)^{\dagger}$, so

$$\begin{bmatrix} A_{11} & [0] \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}^{\dagger} = \begin{bmatrix} {A_{11}}^{*} & \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \\ [0] & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} \end{bmatrix} \begin{bmatrix} A_{11} {A_{11}}^{*} & 0 \\ 0 & \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} + \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} \end{bmatrix}^{\dagger}$$

Since $A_{11} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$ =0.

Thus $M^{\dagger} = \begin{bmatrix} {A_{11}}^{*} \left(A \quad {A_{11}}^{*} \right)^{\dagger} & \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} L^{\dagger} \\ 0 & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} L^{\dagger} \end{bmatrix}$,

Which gives the desired result

Lemma 2.3:

If M is partitioned as in (1.5), then

$M^{\dagger} = \begin{bmatrix} K^{\dagger}A_{11}^{*} & K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \\ 0 & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger} \end{bmatrix}$ where K= $A_{11}^{*}\ A_{11} + \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$ if and only if

$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$=0

Proof:

For the necessity, use (1.1) to obtain $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} K^{\dagger}$ K+ $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$,

which implies $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$=0. For the Sufficiency, again we employ (1.6).

The Pseudo-Inverse of M= $\begin{bmatrix} A_{11} & [0 \quad 0] \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}$ we first determine the Pseudo-Inverse of M given in (1.5).

Theorem:

Let M= $\begin{bmatrix} A_{11} & [0 \quad 0] \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}$.

Then $M^{\dagger} = \begin{bmatrix} K^{+}A_{11}^{*} - K^{+}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}F & K^{+}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} - K^{+}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H \\ F & H \end{bmatrix}$

Where

$$K = A_{11}^{*}A_{11} + \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix},$$

$$D = -A_{11}K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$$

$$E=\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}-\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}$$

$$T=D^{*}D+E^{*}E,$$

$$\mathrm{S}=\mathrm{K}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\left[I-T^{\dagger}T\right],$$

$$\mathrm{F}=\mathrm{T}^{\dagger}\,0^{*}+\left[I-T^{\dagger}T\right]\left[I+S^{*}S\right]^{-1}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{*}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}K^{\dagger}\left[K^{\dagger}A_{11}{}^{*}-K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}T^{\dagger}0^{*}\right],$$

And

$$H=\mathrm{T}^{\dagger}\,E^{*}+\left[I-T^{\dagger}T\right]\left[I+S^{*}S\right]^{-1}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{*}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}K^{\dagger}\left[K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}-K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}T^{\dagger}E^{*}\right]$$

Proof:

Cline [1] has shown that if U V*=U, then

$$\left(U+V\right)^{\dagger}=U^{\dagger}+\left(I-U^{\dagger}V\right)\left[G^{\dagger}+\left(I-G^{\dagger}G\right)\quad QV^{*}\left(U^{\dagger}\right)^{*}\quad U^{\dagger}\left(I-VG^{\dagger}\right)\right],$$

Where $\mathrm{G}=V-UU^{\dagger}V$, $\mathrm{Q}=\left[I+\left(I-G^{\dagger}G\right)\quad V^{*}\left(U^{\dagger}\right)^{*}\quad U^{\dagger}V\left(I-G^{\dagger}G\right)\right]^{-1}$, now,

Let $U=\begin{bmatrix}A_{11} & 0\\ \begin{bmatrix}A_{11}\\ A_{31}\end{bmatrix} & 0\end{bmatrix}$ and $V=\begin{bmatrix}0 & 0\\ 0 & \begin{bmatrix}A_{11} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\end{bmatrix}$

Then $\begin{bmatrix}A_{11} & 0\\ \begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix} & \begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\end{bmatrix}=U+V$

And UV*=0. Hence, clines theorem is applicable by Lemma 2.3, $U^{\dagger}=\begin{bmatrix}K^{\dagger}A_{11}{}^{*} & K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\\ 0 & 0\end{bmatrix},$

Where $\mathrm{K}=\mathrm{A}_{11}{}^{*}A_{11}+\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}$ Thus, $G=\begin{bmatrix}0 & -A_{11}K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\\ 0 & \begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}-\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}K^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\end{bmatrix},$

Let 0= $-A_{11}K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}$ and E= $\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}-\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}$

Then we have G= $\begin{bmatrix}0&D\\0&E\end{bmatrix}$

Therefore, by Lemma 2.1 and the fact $G^{\dagger}=\left[G^{*\dagger}\right]^{*}$, we get $G^{\dagger}=\begin{bmatrix}0&0\\T^{\dagger}D^{*}&T^{\dagger}E^{*}\end{bmatrix}$

Where T= $D^{*}D+E^{*}E$.

Hence, $I-G^{\dagger}G=\begin{bmatrix}I&0\\0&I-T^{\dagger}T\end{bmatrix}$, $U^{\dagger}V\left[I-G^{\dagger}G\right]=\begin{bmatrix}0&K^{\dagger}\begin{bmatrix}A_{22}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}\left(I-T^{\dagger}T\right)\\0&0\end{bmatrix}$

And Q= $\begin{bmatrix}I&0\\0&\left[I+S^{*}S\right]^{-1}\end{bmatrix}$, where S= $K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}\left(I-T^{\dagger}T\right)$, and

$$\mathrm{I}-\mathrm{V}\mathrm{G}^{\dagger}=\begin{bmatrix}I&0\\-\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}T^{\dagger}0^{*}&I-\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}T^{\dagger}E^{*}\end{bmatrix}$$

Now, $U^{\dagger}\left(I-VG^{\dagger}\right)\begin{bmatrix}K^{\dagger}A_{11}{}^{*}-K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}T^{\dagger}D^{*}&K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}-K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}T^{\dagger}E^{*}\\0&0\end{bmatrix}$,

So $G^{\dagger}+\left[I-G^{\dagger}G\right]QV^{*}\left[U^{\dagger}\right]^{*}U^{\dagger}\left[I-VG^{\dagger}\right]=\begin{bmatrix}0&0\\F&H\end{bmatrix}$,

Where

$$F=T^{\dagger}D^{*}+\left(I-T^{\dagger}T\right)\left(I+S^{*}S\right)^{-1}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}^{*}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}K^{\dagger}\left[K^{\dagger}A_{11}{}^{*}-K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}T^{\dagger}D^{*}\right]$$

$$H=T^{\dagger}H^{*}+\left(I-T^{\dagger}T\right)\left(I+S^{*}S\right)^{-1}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}^{*}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}K^{\dagger}\left[K^{\dagger}A_{11}{}^{*}-K^{\dagger}\begin{bmatrix}A_{21}\\A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22}&A_{23}\\A_{32}&A_{33}\end{bmatrix}T^{\dagger}E^{*}\right]$$

And $I - U^{\dagger}V = \begin{bmatrix} I & -K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \\ 0 & I \end{bmatrix}$

Therefore

$$\left(I - U^{\dagger}V\right)\left[G^{\dagger} + \left(I - G^{\dagger}G\right)QV^{*}\left[U^{\dagger}\right]^{*}U^{\dagger}\left(I - VG^{\dagger}\right)\right] =$$

$$\begin{bmatrix} -K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}F & -K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H \\ F & H \end{bmatrix},$$

And finally we get

$$\begin{bmatrix} A_{11} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}^{\dagger}$$

$$= \left(U + V\right)^{\dagger} = \begin{bmatrix} -K^{\dagger}A_{11}{}^{*} - K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}F & -K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} - K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H \\ F & H \end{bmatrix},$$

In [2, p. 748, Theorem 6], c. Meyer has given a formula for (1)-inverses of partitioned upper block triangular matrices. Our theorem also accomplishes this task, since the Pseduo is clearly a (1)-inverse. However, since (1)-inverses are not unique, our results are, in general, different from those of Meyer. For example, if

At this point, we note the following identities, whose proofs are straight forward.

$$T = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} E \qquad \ldots(3.1)$$

If $\mathrm{R} = \mathrm{I} + \mathrm{S}^{*}S$, then $T^{\dagger}TR^{-1} = R^{-1}T^{\dagger}T$ …(3.2)

$$D^{*}A_{11} + E^{*}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = 0 \qquad \ldots(3.3)$$

$$\mathrm{F} = T^{\dagger}D^{*} + \mathrm{R}^{-1}S^{*}\left[K^{\dagger}A_{11}{}^{*} - K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}T^{\dagger}D^{*}\right] \qquad \ldots(3.4)$$

$$H = T^{\dagger}E^{*} + \mathrm{R}^{-1}S^{*}\left[K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} - K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}T^{\dagger}E^{*}\right] \qquad \ldots(3.5)$$

We shall assume throughout the remainder of the paper that M is pseudo partitioned as in (1.5) moreover, we now consider necessary and sufficient conditions for $M^{\dagger}$ to be upper block triangular, lower block triangular, and list at the end of the paper some special for us.

Corollary 3.6

$M^{\dagger}=\begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$ and only if $S^{*}K^{\dagger}A_{11}^{\ *}=0$ and $D^{*}=0$, where S, K, and D are as defined in the theorem,

Proof:

From the Theorem, we can see that $M^{\dagger}=\begin{bmatrix} X & Y \\ O & Z \end{bmatrix} \Leftrightarrow F=0$.

But from (3.4), we have

$$RF=RT^{\dagger}D^{*}+S^{*}K^{\dagger}A_{11}^{\ *}-S^{*}K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}T^{\dagger}D^{*} \qquad \ldots(3.7)$$

By the definition of F, we have

$TF=TT^{\dagger}D^{*}=D^{*}$ since $N(T)\subset N(\cap)$

Thus F=0 implies D^{*}=0.

From (3.7), we get, $S^{*}K^{\dagger}A_{11}^{\ *}=0$ and $D^{*}=0 \Leftrightarrow F=0$

This completes the Proof

Note.

$$F=0 \Rightarrow T=E^{*}E \Rightarrow T^{\dagger}E^{*}=E^{\dagger} \Rightarrow H=E^{\dagger}+\mathrm{R}^{-1}S^{*}\left[K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}-K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}E^{\dagger}\right]$$

Corollary 3.8 [2, p.746 Theorem 4]

$$M^{\dagger}=\begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$$

If and only if $N(A_{11})\subset N\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$ and $N\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*}\subset N\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}$. In this case, we have

$$M^{\dagger}=\begin{bmatrix} A_{11}^{\ \dagger} & 0 \\ -\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}A_{11}^{\ \dagger} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \end{bmatrix}$$

Proof from the theorem, we see that

$$M^{\dagger}=\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$$

If and only if $K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}=K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H$. If $K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}=K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H$,

Then $KK^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}=KK^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H$, and we have

$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}=\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H$. Now $TH=TT^{\dagger}E^{*}=E^{*}$

Implies $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*}EH=E^{*}$. Thus $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H=\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*}$ and using(1.6)

We get $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H=\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger}$, and $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H=\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger}$, hence

$$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}=\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H=\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger},$$

And $N\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*}\subseteq N\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$,

It can be shown that $H=T^{\dagger}E^{*}$ and $F=T^{\dagger}D^{*}$ If $K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}=K^{\dagger}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}H$.

Thus

$$FA_{11}+H\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}=T^{\dagger}D^{*}A_{11}+T^{\dagger}E^{*}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}=T^{\dagger}\left[D^{*}A_{11}+E^{*}\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\right]=0$$

By (3.3), and we get

$$FA_{11}=-H\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \qquad \ldots(3.9)$$

Note next that

$$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} A_{11}^{*} A_{11} - \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} F$$

$$A_{11} = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} A_{11}^{*} A_{11} - \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \left[-H \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \right] \text{ by (3.9)}$$

This last term is the same as $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} K = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$

Finally,

$$\left[\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} A_{11}^{*} A_{11} - \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} K^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} F A_{11} \right] A_{11}^{\dagger} A_{11} = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} A_{11}^{\dagger} A_{11} \text{ yields}$$

$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} A_{11}^{\dagger} A_{11}$, which is equivalent to $N(A_{11}) \subseteq \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$. On the other hand, it is straight forward to verify that when $N(A_{11}) \subseteq N \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$ and

$N \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{*} \subseteq N \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$, then $M^{\dagger} = \begin{bmatrix} A_{11}^{\dagger} & 0 \\ -\begin{pmatrix} A_{22} & A^{23} \\ A_{32} & A_{33} \end{pmatrix}^{\dagger} \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} A_{11}^{\dagger} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{\dagger} \end{bmatrix}$

We note that if m is invertible (ie A_{11} and $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$ are invertible), then

$$M^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} A_{11}^{-1} & \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \end{bmatrix}$$

Suppose $A_{11} = 0$. Then $M^{\dagger}$ is lower block triangular if and only if $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = 0$. There are many special cases which can be derived from corollary 3.8.

In conclusion, the following special results can be obtained.

If $K^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} = K^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}^{*} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} H$, then $K^{\dagger} A_{11}^{*} - K^{\dagger} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} F = A_{11}^{\dagger}$,

$$F=-\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}A_{11}{}^{\dagger},\ H\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}=T^{\dagger}T,\ H=\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}, \text{ and } S=0 \quad \ldots(3.10)$$

$$M^{\dagger}=\begin{bmatrix}A_{11}{}^{\dagger} & \begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{\dagger}\\ 0 & \begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\end{bmatrix} \text{ if and only if } \begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}=0 \text{ and } A_{11}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}=0 \ldots(3.11)$$

$$M^{\dagger}=\begin{bmatrix}A_{11}{}^{\dagger} & D^{\dagger}\\ 0 & \begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}-\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}D^{\dagger}\end{bmatrix},$$

where $D=\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}-\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}$

If and only if $A_{11}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}=0$ and $\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}=\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}D^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}\ldots(3.12)$

$$M^{\dagger}=\begin{bmatrix}A_{11}{}^{\dagger} & \rho\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\\ 0 & \begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}-\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}\rho\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\end{bmatrix}$$

Where $\rho=Q^{-1}\left[\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}^{*}\right]$ and $Q=I+\left[\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}\right]^{*}\left[\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{\dagger}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}\right]$

If and only if $A_{11}\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}=0$ and $N\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}^{*}\subset N\begin{bmatrix}A_{21}\\ A_{31}\end{bmatrix}$ …(3.13)

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Author for Correspondence: B.K.N. Muthugobal

Bharathidasan University constituent Arts and Science College, Nannilam.
Tamil Nadu – India.