Journal of Chemical, Biological and Physical Sciences

An International Peer Review E-3 Journal of Sciences

*Available online at* www.jcbsc.org

*Section C: Physical Sciences*

CODEN (USA): JCBPAT

Research Article

# A New Sub-Equation Method and Novel Interaction Solutions of the Boussinesq Equation

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**Received:** 08 December 2015; **Revised:** 19 December 2015; **Accepted:** 4 January 2016

**Abstract:** In this paper, a new auxiliary equation method is presented. Analytical multiple function solutions including trigonometric function, exponential function, hyperbolic function and other functions can be easily obtained. Novel exact interaction solutions of the Boussinesq equation are obtained successfully by using this new auxiliary equation method. It is very significant to help physicists to analyze special phenomena in their relevant fields accurately.

**Keywords:** travelling wave, soliton, interaction solution, Boussinesq equation

AMS Subject Classification: 35Q51, 35Q53

## INTRODUCTION

As we known, the complicated nature phenomena are often well described by nonlinear partial differential equations. The most representative nonlinear equation is the Boussinesq equation.

$$u_{tt} - \alpha u_{xx} - \beta (u^2)_{xx} - \gamma u_{xxxx} = 0 \quad , \qquad \textbf{(1)}$$

Where *α, β, γ* are arbitrary constants, and u = u(x,t) is a sufficiently often differentiable function denoting velocity averaged over depth. The above model was original introduced by Boussinesq to describe the propagation of long waves in shallow water[1, 2].

Soliton solutions of nonlinear partial differential equations have important applications in nonlinear optics, theoretical physics, plasma physics, fluid dynamics, semiconductors and other fields. It is meaningful to solve various exact solutions including trigonometric function, exponential function, hyperbolic function and other functions. The investigation of such analytical solutions helps us to understand the complicated physics phenomena well. In the past decades, many methods are proposed to obtain exact solutions of nonlinear partial differential equations: such as inverse scattering theory[3], Hirotas bilinear method[4], the truncated Painlev expansion[5], Darboux transformation[6] and so on. In recent years, a large number of powerful methods to solve nonlinear partial differential equations are considered. One of important methods is the auxiliary equation method, it includes the homogeneous balance method[7], sine-cosine method[8], the sech-function method[9], the hyperbolic tangent function method[10, 11], the multiple exp-function method[12,13] , the expansion method[14], the generalize method[15], they all are collectively known as the auxiliary equation method.

It has attracted extensive attention as it's concise and understandable. Ma[16] and Chen[17-24] are devoted to constructing special interaction soliton solutions by using combination of auxiliary equations and get great success. But the solutions of the solvable auxiliary equation are singular soliton solutions. In this paper, this novel auxiliary equation can successfully to obtain multiple function solutions which including trigonometric function, exponential function, hyperbolic function and other functions.

This paper is organized as follow: a new auxiliary equation which we found out multiple function solutions in section 2. In section 3, we introduce this new auxiliary equation method. It can be applied to many nonlinear partial differential equations in different fields effectively. In section 4, this method is applied to the Boussinesq equation successfully. Many new exact interaction solutions are obtained. Some conclusions and discussions are given in section 5.

**The new solutions of the novel sub-equation:**

For the novel auxiliary equation reads:

$$\phi'' = P + Q\phi + R\phi^3 \ , \tag{2}$$

Where $\phi'' = \phi''(\xi)$ .If P = R = 0, Q = 1, we obtain new multiple solutions of eq. (2) in the following:

$$\phi_1 = \frac{(\tanh(\xi)+\tan(\xi))e^{\xi}}{1+\tan(\xi)\coth(\xi)+\tanh(\xi)+\tan(\xi)}, \tag{3}$$

$$\phi_2 = \frac{(\tanh(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\tan(\xi)+\tan(\xi)+\coth(\xi)}, \tag{4}$$

$$\phi_3 = \frac{(\tanh(\xi)+\cot(\xi))e^{\xi}}{1+\cot(\xi)\coth(\xi)+\tanh(\xi)+\cot(\xi)}, \tag{5}$$

$$\phi_4 = \frac{(\cot(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\cot(\xi)+\cot(\xi)+\coth(\xi)}, \tag{6}$$

$$\phi_5 = \frac{(1+\tan(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\tan(\xi)+\tan(\xi)\coth(\xi)}, \tag{7}$$

$$\phi_6 = \frac{(1+\cot(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\cot(\xi)+\cot(\xi)\coth(\xi)}, \quad \mathbf{(8)}$$

$$\phi_7 = \frac{(1+\tanh(\xi)\tan(\xi))e^{\xi}}{1+\tan(\xi)+\coth(\xi)+\tanh(\xi)\tan(\xi)}, \quad \mathbf{(9)}$$

$$\phi_8 = \frac{(1+\tanh(\xi)\cot(\xi))e^{\xi}}{1+\cot(\xi)+\coth(\xi)+\tanh(\xi)\cot(\xi)}, \quad \mathbf{(10)}$$

If we let $f(\xi) = \frac{1}{\ln(\varphi(\xi))}$, then we get:

$$f'' = 2f^3, \quad (11)$$

Where $f'' = f''(\xi)$. We obtain new multiple solutions of eq. (11) in the following:

$$f_1 = \{\ln[\frac{(\tanh(\xi)+\tan(\xi))e^{\xi}}{1+\tan(\xi)\coth(\xi)+\tanh(\xi)+\tan(\xi)}]\}^{-1}, \quad \mathbf{(12)}$$

$$f_2 = \{\ln[\frac{(\tanh(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\tan(\xi)+\tan(\xi)+\coth(\xi)}]\}^{-1}, \quad \mathbf{(13)}$$

$$f_3 = \{\ln[\frac{(\tanh(\xi)+\cot(\xi))e^{\xi}}{1+\cot(\xi)\coth(\xi)+\tanh(\xi)+\cot(\xi)}]\}^{-1}, \quad \mathbf{(14)}$$

$$f_4 = \{\ln[\frac{(\cot(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\cot(\xi)+\cot(\xi)+\coth(\xi)}]\}^{-1}, \quad \mathbf{(15)}$$

$$f_5 = \{\ln[\frac{(1+\tan(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\tan(\xi)+\tan(\xi)\coth(\xi)}]\}^{-1}, \quad \mathbf{(16)}$$

$$f_6 = \{\ln[\frac{(1+\cot(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\cot(\xi)+\cot(\xi)\coth(\xi)}]\}^{-1}, \quad \mathbf{(17)}$$

$$f_7 = \{\ln[\frac{(1+\tanh(\xi)\tan(\xi))e^{\xi}}{1+\tan(\xi)+\coth(\xi)+\tanh(\xi)\tan(\xi)}]\}^{-1}, \quad \mathbf{(18)}$$

$$f_8 = \{\ln[\frac{(1+\tanh(\xi)\cot(\xi))e^{\xi}}{1+\cot(\xi)+\coth(\xi)+\tanh(\xi)\cot(\xi)}]\}^{-1}, \quad \mathbf{(19)}$$

Some imaginary solutions are omitted above，therefore real interaction solutions of nonlinear partial deferential equations are obtained when we apply this auxiliary equation into nonlinear partial deferential equations.

**The new sub-equation method:**

**Step1:** For a given nonlinear partial differential equation with independent variables x,t,...:

$$P(t,x,u_t,u_x,u_{xt},u_{xx}....)=0 \quad (20)$$

We make a transformation as follow:

$$u(x,t)=u(\xi),\xi=k(x-ct), \quad (21)$$

Where k is the wave number, and c is the wave velocity.

**Step2:** Inserting eq. (21) into Eq. (20), we get an ordinary differential equation:

$$P(k,c,u,u'',u'...)=0 \quad (22)$$

**Step3:** We assume exact solutions of eq. (22) in the following form:

$$u(\xi)=\sum_{i=0}^{n} a_i f^i(\xi) \quad (23)$$

Where n is a positive integer determined by the balance principle in eq. (22). $f(\xi)$ satisfies the auxiliary eq.(11). Substituting (23) into (22). We obtain a set of algebra equations when we set each coefficients of $f^i(\xi)$ to zeroes. Therefore $a_i$ will be determined by solving the set of algebra equations. We will apply the method to the Boussinesq equation.

**New Interaction Solutions of the Boussinesq Equation:**

The Boussinesq Equation is:

$$u_{tt}-\alpha u_{xx}-\beta(u^2)_{xx}-\gamma u_{xxxx}=0, \quad (24)$$

Where $\alpha,\beta,\gamma$ are constants.

We assume the solution of eq. (24) as the following,

$$u(\xi)=\sum_{i=0}^{n} a_i f^i(\xi),\xi=k(x-ct), \quad (25)$$

Where n is positive integer and equate to 2.It is determined by balancing the linear term of $u_{xxxx}$ and the nonlinear term of $(u^2)_{xx}$. $f(\zeta)$ satisfies eq.(11). For the sake of simplicity, we take the exact solutions of eq.(24) as following form:

$$u(\xi)=a_0+a_1 f(\xi)+a_2 f^2(\xi),\xi=k(x-ct), \quad (26)$$

Where $a_0, a_1, a_2$, k, c are constants, they could be all determined in the later. Hence, substituting eq.(26) into eq.(24) along with aid of the auxiliary equation and equating the coefficients of $f^j(\xi)(0 \le j \le 4)$ to zeroes, a set of algebraic equations are obtained, so that the unknown parameters $a_0, a_1, a_2$, k, c can be solved by using the computation with Maple.

We get the following result:

$$a_0 = \frac{c^2+\alpha}{2c\beta}, a_1 = \frac{2k^2c\gamma}{\beta}, a_2 = 0$$
,

We obtain interaction solutions of eq.(24) in the following:

$$u_1 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(\tanh(\xi)+\tan(\xi))e^{\xi}}{1+\tan(\xi)\coth(\xi)+\tanh(\xi)+\tan(\xi)}]\}^{-1}, \quad (27)$$

$$u_2 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(\tanh(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\tan(\xi)+\tan(\xi)+\coth(\xi)}]\}^{-1}, \quad (28)$$

$$u_3 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(\tanh(\xi)+\cot(\xi))e^{\xi}}{1+\cot(\xi)\coth(\xi)+\tanh(\xi)+\cot(\xi)}]\}^{-1}, \quad (29)$$

$$u_4 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(\cot(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\cot(\xi)+\cot(\xi)+\coth(\xi)}]\}^{-1}, \quad (30)$$

$$u_5 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(1+\tan(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\tan(\xi)+\tan(\xi)\coth(\xi)}]\}^{-1}, \quad (31)$$

$$u_6 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(1+\cot(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\cot(\xi)+\cot(\xi)\coth(\xi)}]\}^{-1}, \quad (32)$$

$$u_7 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(1+\tanh(\xi)\tan(\xi))e^{\xi}}{1+\tan(\xi)+\coth(\xi)+\tanh(\xi)\tan(\xi)}]\}^{-1}, \quad (33)$$

$$u_8 = \frac{c^2+\alpha}{2c\beta} + \frac{2k^2c\gamma}{\beta}\{\ln[\frac{(1+\tanh(\xi)\cot(\xi))e^{\xi}}{1+\cot(\xi)+\coth(\xi)+\tanh(\xi)\cot(\xi)}]\}^{-1}, \quad (34)$$

Where $\zeta = k(x-ct)$. k and c are arbitrary nonzero constants.

We get solutions eqs(27)-(34) which contain trigonometric function, exponential function and hyperbolic functions. They are all determined by solvable new arbitrary equation and they are novel interaction solutions of eq.(24) which are not obtained in ref[1, 2]. The phenomena of appearance of interaction waves are instantaneous and changeful. The interaction solutions are so complicated that the influences of the solutions are not easy to uncover. The effects are so significant, as nonlinear phenomena appear always everywhere in nature.

## CONCLUSION AND DISCUSSION

In this paper, a new auxiliary equation is considered which we seek out multiple function solutions. A new auxiliary equation method is presented. Some new exact interaction solutions of the Boussinesq equation are obtained by using this method. This method can be easily and effectively applied to other partial differential equations. It draws great attention that solutions of the novel auxiliary equation themselves include trigonometric function, hyperbolic function, exponential function and other functions. It is not proposed in previous auxiliary equation methods. Complicate physical phenomena in nonlinear model systems will be described well when we analyze the typical interaction solutions we obtained in this paper.

## ACKNOWLEDGMENT

This research is supported by Applied Mathematics Enhancement Program of Linyi University, the Natural Science Foundation of Shandong Province of China under Grant (ZR2010AM014) and the Scientific Research Fund from Shandong Provincial Education Department (J07WH04).

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