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Radiation-Type of Problems Using Generalized Hankel Type Integral Transformation

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Abstract: In this paper, the generalized Hankel type integral transformation of three real parameters is used to solve boundary value problems. Three problems of heat conduction in an infinite, semi-infinite and a finite circular cylinder bounded by given surfaces with radiation-type boundary value conditions are solved using the transforms.

Keywords: Boundary conditions, generalized Hankel type integral transformation, radiation-type, differential equations.

INTRODUCTION

The generalization of the classical Hankel transformation is given as

$$\int_0^\infty \sqrt{xy}\, J_\mu(xy)\, f(x)\, dx \, ; \mu \geq -1/2 \tag{1.1}$$

By the many authors. Authors[8], have solved generalized heat conduction equation with radiation-type boundary conditions, under the given initial temperature using finite domain.

In this paper, the author applied the generalized Hankel type integral transformation of three real parameters is used to solve boundary value problems. Three problems of heat conduction in an infinite, semi-infinite and a finite circular cylinder bounded by given surfaces with radiation-type boundary value conditions are solved using the transforms, in an infinite domain with the existing theory developed by Malgonde and Bhandewar[6].

PRELIMINARY RESULTS

Malgonde[1] introduced the generalized Hankel type transformation depending on three real parameters (α,β,ν) and it is represented in this paper as

$$F_1(y)=F_{1,\alpha,\beta,\nu,\mu}[f(x)](y)=\beta^2\nu^2 y^{2\nu-2\alpha-1}\int_0^\infty \mathcal{J}_{\alpha,\beta,\nu,\mu}(xy)f(x)\,dx;\ \mu\geq -2 \tag{2.1}$$

where α,β and ν are any arbitrary real italic numbers and $J_\mu(z)$ is the Bessel function of the first kind of order μ.

Here $\mathcal{J}_{\alpha,\beta,\nu,\mu}(xy)=(xy)^\alpha J_\mu\left(\beta(xy)^\nu\right)$; $\Upsilon_{\alpha,\beta,\nu,\mu}(xy)=(xy)^\alpha Y_\mu\left(\beta(xy)^\nu\right)$. $Y_{\alpha,\beta,\nu,\mu}(\lambda_n x)$ is the Bessel-function of the second kind of order μ. The operator is defined by $\Delta_{\alpha,\beta,\nu,\mu}=x^{-\mu\nu+\alpha+1-2\nu}Dx^{2\mu\nu+1}Dx^{-\mu\nu-\alpha}$ and $\Delta^*_{\alpha,\beta,\nu,\mu}=x^{-\mu\nu-\alpha}Dx^{2\mu\nu+1}Dx^{-\mu\nu+\alpha+1-2\nu}$.

$$\begin{aligned}&\beta^2\nu^2 y^{2\nu-2\alpha-1}\int_0^R \mathcal{J}_{\alpha,\beta,\nu,\mu}(ty)\mathcal{J}_{\alpha,\beta,\nu,\mu}(xy)\,dy\\ &=\frac{R^\nu}{x^{2\nu}-t^{2\nu}}\left[x^\nu J_{\mu+1}\left(\beta(xR)^\nu\right)J_\mu\left(\beta(tR)^\nu\right)-t^\nu J_{\mu+1}\left(\beta(tR)^\nu\right)J_\mu\left(\beta(xR)^\nu\right)\right]\end{aligned} \tag{2.2}$$

asymptotic representations of Bessel functions enable that for any $a>0$ in[3].

Lemma 1. (Inversion Formula) If $f(x)\in L_1(0,\infty)$, if $f(x)$ is of bounded variation in a neighbourhood of the point $x=x_0>0$, if $\mu\geq-\frac{1}{2}$, if $F_1(y)$ is defined as the conventional Hankel transformation and $\int_0^\infty |f(x)|x^{\alpha-\nu/2}dx$ exists as[5], then

$$h_\mu^{-1}F=\lim_{R\to\infty} x_0^{2\nu-2\alpha-1}\int_0^\infty \mathcal{J}_{\alpha,\beta,\nu,\mu}(xy)F_1(y)dy=\frac{1}{2}\left[f(x_0+0)+f(x_0-0)\right]. \tag{2.3}$$

Note that $\mu\geq-\frac{1}{2}$, the conventional inverse Hankel transformation h_μ^{-1} is defined by precisely the same formula as is the direct Hankel transformation h_μ; in symbols, $h_\mu=h_\mu^{-1}$.

Lemma 2. (Mixed Parseval's Equation) If $f(x)x^{\alpha+\mu}$ and $F_2(y)y^{\mu\nu-\alpha-1+2\nu}$ are in $L_1(0,\infty)$, $F_1(y)=F_{1,\alpha,\beta,\nu,\mu}\left[f(x)\right]y$ and

$$F_2(y)=F_{2,\alpha,\beta,\nu,\mu}[g(x)](y)=\beta^2\nu^2\int_0^\infty x^{2\nu-2\alpha-1}\mathcal{J}_{\alpha,\beta,\nu,\mu}(xy)f(x)dx;\ \mu\geq -1/2 \tag{2.4}$$

$$\int_0^\infty f(x)g(x)dx=\int_0^\infty F_1(y)F_2(y)dy.$$

An operation-transform formula was introduced in solving certain differential equations involing generalized functions[7], where

$$\left\langle\left(\Delta^*_{\alpha,\beta,\nu,\mu}\right)^k f(x),\phi(x)\right\rangle=\left\langle f(x),\Delta^k_{\alpha,\beta,\nu,\mu}\phi(x)\right\rangle \tag{2.5}$$

For each $k=1,2,3,\ldots$.

$$\begin{aligned}\Delta^*_{\alpha,\beta,\nu,\mu}&=x^{-\mu\nu-\alpha}Dx^{2\mu\nu+1}Dx^{-\mu\nu+\alpha+1-2\nu}\\&=x^{2-2\nu}D^2-(4\nu-2\alpha-3)x^{1-2\nu}D-\left[\mu^2\nu^2-(\alpha+1-2\nu)^2\right]x^{-2\nu}.\end{aligned} \tag{2.6}$$

The theorem was established to enable to transform differential equations of the form

$$P\left[\Delta^*_{\alpha,\beta,\nu,\mu}\right]u=g, \tag{2.7}$$

where u and g possess $F'_{1,\alpha,\beta,\nu,\mu}$-transforms and P is any polynomial having no zeros on $-\infty<x\leq 0$ into an algebraic equation of the form

$$p\left[-\nu^2\beta^2y^{2\nu}\right]U(y)=G(y), \tag{2.8}$$

Where $U(y)=\left(F'_{1,\alpha,\beta,\nu,\mu}u(x)\right)(y)$ and $G(y)=\left(F'_{1,\alpha,\beta,\nu,\mu}g(x)\right)(y)$.

$$F'_{1,\alpha,\beta,\nu,\mu}\left[\Delta^{*^k}_{\alpha,\beta,\nu,\mu}f\right](y)=\left(-\beta^2\nu^2\right)^k y^{2\nu k}F'_{1,\alpha,\beta,\nu,\mu}y \tag{2.9}$$

For every $f\in H'_{\alpha,\beta,\nu,\mu}(\sigma_f)$.

A large number of differential equations taking place in the formulation of problems of physics and engineering are specialization of the following generalized Cauchy's problem to determine the function $u(x,t)$ which satisfies

$$\Delta^*_{\alpha,\beta,\nu,\mu}u-\frac{1}{c^2}\frac{\partial^2u}{\partial t^2}=0, \tag{2.10}$$

For $\mu\geq -1/2$, α,β are real numbers and $c>0$.

The Hankel transform arises naturally in the problems posed in the cylindrical coordinates which when solved by using technique of separation of variables, involve Bessel functions. Generalized Hankel type transformation is more appropriate in solving differential equations with boundary conditions in which there is an axial symmetry. To deal with the problems formulated in the infinite domain, integral transforms are introduced.

Applications: Throughout these examples, ρ can be chosen as large as possible.

a) Heat conduction in an infinite cylinder: To solve the following generalized heat conduction equation with radiation-type boundary conditions, under the given initial temperature[1]:

$$\frac{1}{C^2}\frac{\partial u}{\partial t}=r^{2-2\nu}\left(\frac{\partial^2 u}{\partial r^2}-\frac{(4\nu-2\alpha-3)}{r}\frac{\partial u}{\partial r}-\frac{\left(\mu^2\nu^2-(\alpha+1-2\nu)^2\right)}{r^2}u\right), \tag{3.1}$$

where $0<r\le\infty, t>0$ and $u(r,t)$, denotes the temperature at any radial position (r,z) at time $t;C$ is a constant that depends on the material of the cylinder.

The initial and boundary conditions are:

$$u(r,0)=f(r),\ 0<r\le\infty, \tag{3.2}$$

$$h_{1,\alpha,\beta,\nu,\mu}u+\left.\frac{\partial u}{\partial r}\right|_{r=0}=g_1(t),\ h_{2,\alpha,\beta,\nu,\mu}u+\left.\frac{\partial u}{\partial r}\right|_{r=\rho}=g_2(t);\ t>0. \tag{3.3}$$

Taking generalized Hankel type transformation defined by (2.1) with respect to variable $'r'$ of eq. (3.1) and using the property (2.9) as in[4] to get

$$\frac{1}{C^2}\frac{\partial U}{\partial t}=\rho^{2\nu-2\alpha-1}\left[h_{2,\alpha,\beta,\nu,\mu}\rho+u'(\rho)\right]N_\mu(\lambda_n\rho)-\nu^2\beta^2\lambda_n^{2\nu}U$$

where $N_\mu(\lambda_n\rho)=\dfrac{R^\nu}{\rho^{2\nu}-\lambda_n^{2\nu}}\left[\rho^\nu J_{\mu+1}\left(\beta(\rho R)^\nu\right)J_\mu\left(\beta(\lambda_n R)^\nu\right)-\lambda_n^{2\nu}J_{\mu+1}\left(\beta(\lambda_n R)^\nu\right)J_\mu\left(\beta(\rho R)^\nu\right)\right].$

Thus the differential equation becomes

$$\frac{\partial U}{\partial t}+\nu^2\beta^2C^2\lambda_n^{2\nu}U=C^2\rho^{2\nu-2\alpha-1}N_\mu(\rho\lambda_n)g_2(t). \tag{3.4}$$

Its solution is given by

$$U(\lambda_n,t)=e^{-\nu^2\beta^2C^2\lambda_n^{2\nu}t}C^2\rho^{2\nu-2\alpha-1}\left[\int_0^\rho e^{\nu^2\beta^2C^2\lambda_n^{2\nu}x}N_\mu(\rho\lambda_n)g_2(x)\,dx\right]+\bar{f}(\lambda_n). \tag{3.5}$$

Here $\bar{f}(\lambda_n)$ is the transform of the initial condition and is defined as

$$U(\lambda_n,0)=\bar{f}(\lambda_n).$$

b) Heat conduction in a semi-infinite cylinder: To solve the generalized heat conduction equation with radiation-type boundary conditions, under the given initial temperature[8], using generalized Hankel type transformation:

$$\frac{1}{C^2}\frac{\partial u}{\partial t}=r^{2-2\nu}\left(\frac{\partial^2 u}{\partial r^2}-\frac{(4\nu-2\alpha-3)}{r}\frac{\partial u}{\partial r}-\frac{\left(\mu^2\nu^2-(\alpha+1-2\nu)^2\right)}{r^2}u\right)+\frac{\partial^2 u}{\partial z^2} \tag{3.6}$$

where $0<r\le\infty, t>0$ and $u(r,z,t)$, denotes the temperature at any radial position (r,z) at time $t;C$ is a constant that depends on the material of the cylinder.

The initial and boundary conditions are:

$$u(r,z,0)=f(r,z),\ 0<r\le\infty, \tag{3.7}$$

$$u(r,0,t)=I(r,t),\ \lim_{z\to\infty}u(r,z,t)=0, \tag{3.8}$$

$$h_{1,\alpha,\beta,\nu,\mu}u+\left.\frac{\partial u}{\partial r}\right|_{r=0}=g_1(z,t), h_{2,\alpha,\beta,\nu,\mu}u+\left.\frac{\partial u}{\partial r}\right|_{r=\rho}=g_2(z,t); t>0, \rho\to\infty.$$

Taking generalized Hankel type transformation defined by (2.1) with respect to variable r of eq. (3.6) and using the property (2.9) to get

$$\frac{1}{C^2}\frac{\partial U}{\partial t}=\rho^{2\nu-2\alpha-1}\left[h_{2,\alpha,\beta,\nu,\mu}\rho+u'(\rho)\right]N_\mu(\lambda_n\rho)-\nu^2\beta^2\lambda_n^{2\nu}U+\frac{\partial^2 U}{\partial z^2}.$$

By using boundary conditions, equation becomes[2]

$$\frac{1}{C^2}\frac{\partial U}{\partial t}=\rho^{2\nu-2\alpha-1}g_2(z,t)N_\mu(\lambda_n\rho)-\nu^2\beta^2\lambda_n^{2\nu}U+\frac{\partial^2 U}{\partial z^2}.$$

$$\frac{\partial U}{\partial t}+\nu^2\beta^2C^2\lambda_n^{2\nu}U=C^2\rho^{2\nu-2\alpha-1}N_\mu(\lambda_n\rho)g_2(p,t)+\frac{\partial^2 U}{\partial z^2}. \tag{3.9}$$

Its solution is given by

$$U(\lambda_n,p,t)=e^{-\nu^2\beta^2C^2\lambda_n^{2\nu}t}C^2\rho^{2\nu-2\alpha-1}\left[\int_0^t e^{\nu^2\beta^2C^2\lambda_n^{2\nu}x}\left\{N_\mu(\rho\lambda_n)g_2(p,x)+p\bar{I}(\lambda_n,x)\right\}dx+\bar{f}(\lambda_n,p)\right]\ldots.. \tag{3.10}$$

c) Heat conduction in a finite cylinder: To solve the generalized heat conduction equation with radiation-type boundary conditions, in a finite cylinder $z_1<z<z_2$, under the given initial temperature[8], using generalized Hankel type transformation:

$$\frac{1}{C^2}\frac{\partial u}{\partial t}=r^{2-2\nu}\left(\frac{\partial^2 u}{\partial r^2}-\frac{(4\nu-2\alpha-3)}{r}\frac{\partial u}{\partial r}-\frac{\left(\mu^2\nu^2-(\alpha+1-2\nu)^2\right)}{r^2}u\right)+\frac{\partial^2 u}{\partial z^2} \tag{3.11}$$

Where $0<r\le\infty, t>0$ and $u(r,z,t)$, denotes the temperature at any radial position (r,z) at time $t;C$ is a constant that depends on the material of the cylinder.

The initial and boundary conditions are:

$$u(r,z,0)=f(r,z),\ 0<r\le\infty, \tag{3.12}$$

$$u(r,z_1,t)=I_1(r,t), u(r,z_2,t)=I_2(r,t), \tag{3.13}$$

$$h_{1,\alpha,\beta,\nu,\mu}u+\left.\frac{\partial u}{\partial r}\right|_{r=0}=g_1(z,t), h_{2,\alpha,\beta,\nu,\mu}u+\left.\frac{\partial u}{\partial r}\right|_{r=\rho}=g_2(z,t); t>0, \rho\to\infty.$$

Taking generalized Hankel type transformation defined by (2.1) with respect to variable $'r'$ of eq. (3.11) and using the property (2.9) to get

$$\frac{1}{C^2}\frac{\partial U}{\partial t}=\rho^{2\nu-2\alpha-1}\left[h_{2,\alpha,\beta,\nu,\mu}\rho+u'(\rho)\right]N_\mu(\lambda_n\rho)-\nu^2\beta^2\lambda_n^{2\nu}U+\frac{\partial^2 U}{\partial z^2}.$$

By using boundary conditions, equation becomes

$$\frac{1}{C^2}\frac{\partial U}{\partial t}=\rho^{2\nu-2\alpha-1}g_2(z,t)N_\mu(\lambda_n\rho)-\nu^2\beta^2\lambda_n^{2\nu}U+\frac{\partial^2 U}{\partial z^2}.$$

$$\frac{\partial U}{\partial t}+\nu^2\beta^2C^2\lambda_n^{2\nu}U=C^2\rho^{2\nu-2\alpha-1}N_\mu(\lambda_n\rho)g_2(p,t)+\frac{\partial^2 U}{\partial z^2}. \tag{3.14}$$

Its solution is given by

$$U(\lambda_n,p,t)=e^{-\nu^2\beta^2C^2\lambda_n^{2\nu}t}C^2\rho^{2\nu-2\alpha-1}\left[\int_0^t e^{\nu^2\beta^2C^2\lambda_n^{2\nu}x}\left\{N_\mu(\rho\lambda_n)g_2(p,x)+p\bar{I}_1(\lambda_n,x)+p\bar{I}_2(\lambda_n,x)\right\}dx+\bar{f}(\lambda_n,p)\right]. \tag{3.15}$$

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DEDICATIONS

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