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Generalized Fractional Differintegral Operators of the Aleph-Function of two Variables

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Abstract: The object of this article is to study and develop the generalized fractional calculus operators given by Saigo and Maeda [1] for Aleph function of two variables [2]. The results obtained provide unification and extension of the results given by Saxena *et al.*[3-5], Kumar and Choi [6], Ram and Kumar [7]. The results are obtained in compact form and are useful in preparing some tables of operators of fractional calculus. On account of the general nature of the Saigo-Maeda operators and Aleph-function of two variables a large number of new and known results involving fractional calculus operators and several special functions notably H - function of two variables and H - function of two variables follow as special cases of our main findings.

2010 Mathematics Subject Classifications: 26A33, 33E20, 33C45, 33C60, 33C70.

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1. INTRODUCTION

Fractional calculus studies derivatives (and integrals) of non-integer order. It is a classical mathematical field as old as calculus itself [8]. During almost 300 years, fractional calculus was considered as pure mathematics, with nearly no applications. In recent years, however, the situation changed dramatically; with fractional calculus becoming an interesting and useful topic among engineers and applied scientists, and an excellent tool for description of memory and heredity effects [9].One of the trends of the contemporary fractional calculus is the so-called generalized fractional calculus (GFC). Along with the expansion of numerous and even unexpected recent applications of the operators of the classical fractional calculus (FC), the GFC is another powerful tool stimulating the development of this field. It is also generating new classes of special functions (special functions of fractional calculus) and integral transforms, as well as providing new transmutation operators applicable to solve more complicated problems in analysis via their reduction to simpler ones. The GFC poses also: new challenges for interpretations of its operators, similar to the classical fractional integrals and derivatives (see e.g. Podlubny [10] and new open problems for their applications in solving not only theoretical models of fractional (multi-) order differential and integral equations, but mathematical models of real phenomena and events (as it is now well illustrated for the classical FC).

Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$, then the generalized fractional integral operators $I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}$ and $I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}$ of a function $f(x)$ for $\mathrm{Re}(\gamma)>0,$ is defined by Saigo and Maeda [1], in the following form:

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\right) f(t)\,dt, \qquad \text{… (1.1)}$$

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x\right) f(t)\,dt, \qquad \text{… (1.2)}$$

These operators reduce to the fractional integral operators introduced by Saigo [11], due to the following relations:

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x)=I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x), \qquad (\gamma\in\mathbb{C}), \qquad \text{… (1.3)}$$

and

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f(x)=I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f(x), \qquad (\gamma\in\mathbb{C}). \qquad \text{… (1.4)}$$

Let $\alpha,\alpha',\beta,\beta',\gamma\in C$, $\gamma>0$ and $x\in\mathbb{R}_+$, then the generalized fractional differentiation operators [1] involving Appell function F_3 as a kernel can be defined as

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \qquad \text{… (1.5)}$$

$$=\left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f\right)(x) \left(\mathrm{Re}(\gamma)>0;\ n=\left[\mathrm{Re}(\gamma)\right]+1\right) \qquad \text{… (1.6)}$$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \qquad \text{… (1.7)}$$

$$=\left(-\frac{d}{dx}\right)^{n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n}f\right)(x)\left(\mathrm{Re}(\gamma)>0;\ n=\left[\mathrm{Re}(\gamma)\right]+1\right) \quad \ldots (1.8)$$

These operators reduce to Saigo derivative operators [1, 11] as

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x), \quad \left(\mathrm{Re}(\gamma)>0\right); \quad \ldots (1.9)$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x), \quad \left(\mathrm{Re}(\gamma)>0\right). \quad \ldots (1.10)$$

Further [1, p. 394, Eqns. (4.18) and (4.19)] we also have

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=\Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta,\rho+\beta'\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \quad \ldots (1.11)$$

where $\mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\rho)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right]$,

and

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=\Gamma\left[\begin{matrix}1+\alpha+\alpha'-\gamma-\rho,1+\alpha+\beta'-\gamma-\rho,1-\beta-\rho\\ 1-\rho,1+\alpha+\alpha'+\beta'-\gamma-\rho,1+\alpha-\beta-\rho\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \quad \ldots (1.12)$$

where $\mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right]$.

Here, the symbol $\Gamma\left[\begin{matrix}a,b,c\\ d,e,f\end{matrix}\right]$ will be used to represent the ratio of product of gamma functions as

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}.$$

The Aleph-function of two variables defined and given by Saxena *et al.*[2] and studied by Saxena *et al.*[5], Sharma [12]. The Aleph-function of two variables is an extension of the I -function of two variables (see, Sharma and Mishra [13]) which itself is a generalization of G and H -functions of two variables.

The double Mellin-Barnes integral occurring in the present paper will be referred to as the Aleph-function of two variables throughout our present study and will be defined and represented by in the following manner:

$$\aleph[x,y] = \aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|l} x & \left(a_j;\alpha_j,A_j\right)_{1,n},\left[\tau_j\left(a_{ji};\alpha_{ji},A_{ji}\right)\right]_{n+1,p_i}:\left(c_j,C_j\right)_{1,n_1},\left[\tau_j\left(c_{ji'},C_{ji'}\right)\right]_{n_1+1,p_{i'}}; \\ y & \left[\tau_j\left(b_{ji};\beta_{ji},B_{ji}\right)\right]_{1,q_i}:\left(d_j,D_j\right)_{1,m_1},\left[\tau_j\left(d_{ji'},D_{ji'}\right)\right]_{m_2+1,q_{i'}}; \end{array}\right.$$

$$\left.\begin{array}{l} \left(e_j,E_j\right)_{1,n_2},\left[\tau_j\left(e_{ji''},E_{ji''}\right)\right]_{n_2+1,p_{i''}} \\ \left(f_j,F_j\right)_{1,m_2},\left[\tau_j\left(f_{ji''},F_{ji''}\right)\right]_{m_2+1,q_{i''}} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\theta_1(s)\theta_2(\xi)\,x^{-s}y^{-\xi}\,ds\,d\xi, \dots \quad (1.13)$$

where $\omega = \sqrt{-1}$,

and

$$\phi(s,\xi) = \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j s-A_j\xi\right)}{\prod_{i=1}^{r}\tau_i\left\{\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}s+A_{ji}\xi\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}s-B_{ij}\xi\right)\right\}} \quad \dots$$

(1.14)

$$\theta_1(s) = \Omega^{m_1,n_1}_{p_{i'},q_{i'},\tau_{i'};r'}(s) = \frac{\prod_{j=1}^{m_1}\Gamma\left(d_j+D_j s\right)\prod_{j=1}^{n_1}\Gamma\left(1-c_j-C_j s\right)}{\sum_{i'=1}^{r'}\tau_{i'}\left\{\prod_{j=m_1+1}^{q_{i'}}\Gamma\left(1-d_{ji'}-D_{ji'}s\right)\prod_{j=n_1+1}^{p_{i'}}\Gamma\left(c_{ji'}+C_{ji'}s\right)\right\}},$$

$$\theta_2(\xi) = \Omega^{m_2,n_2}_{p_{i''},q_{i''},\tau_{i''};r''}(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma\left(f_j+F_j\xi\right)\prod_{j=1}^{n_2}\Gamma\left(1-e_j-E_j\xi\right)}{\sum_{i''=1}^{r''}\tau_{i''}\left\{\prod_{j=m_2+1}^{q_{i''}}\Gamma\left(1-f_{ji''}-F_{ji''}\xi\right)\prod_{j=n_2+1}^{p_{i''}}\Gamma\left(e_{ji''}+E_{ji''}\xi\right)\right\}},$$

Here, the variables x and y are real or complex different to zero, and an empty product is interpreted as unity. $p_i, p_{i'}, p_{i''}, q_i, q_{i'}, q_{i''}, n, n_1, n_2, m_1, m_2$ are non-negative integers such that $0 \le n \le p_i$, $0 \le n_1 \le p_{i'}$, $0 \le n_2 \le p_{i''}$, $q_i, q_{i'}, q_{i''} > 0$, $\tau_i, \tau_{i'}, \tau_{i''} > 0$ $\left(i = \overline{1,r}, i' = \overline{1,r'}, i'' = \overline{1,r''}\right)$. All the $A's, \alpha's, B's, \beta's, C's, D's, E's$ and $F's$ are assumed to be positive quantities for standardization purpose. The integration path $L_1 = L_{\omega\gamma\infty}\left(\gamma \in \mathbb{R}\right)$ is in the s-plane and extends from $\gamma - \omega\infty$ to $\gamma + \omega\infty$ with loops, and is such that the poles of $\Gamma\left(d_j + D_j s\right), j = \overline{1,m_1}$ (the symbol $\left(\overline{1,m_1}\right)$ is used for $1,2,\dots,m_1$) lies to the right, and the poles of $\Gamma\left(1-c_j-C_j s\right), j = \overline{1,n_1}$,

$\Gamma\left(1-a_j-\alpha_j s+A_j\xi\right), j=\overline{1,n}$ to the left of the contour. The integration path $L_2=L_{\omega\gamma\infty}\left(\gamma\in\mathbb{R}\right)$ is in the ξ -plane and extends from $\gamma-\omega\infty$ to $\gamma+\omega\infty$ with loops, and is such that the poles of $\Gamma\left(f_j+F_j\xi\right), j=\overline{1,m_2}$ lies to the right, and the poles of $\Gamma\left(1-e_j-E_j\xi\right), j=\overline{1,n_2}$, $\Gamma\left(1-a_j-\alpha_j s+A_j\xi\right), j=\overline{1,n}$ to the left of the contour.

The existence conditions for the defining integral (1.13) are given below:

$$\left|\arg(x)\right|<\frac{\pi}{2}\Xi,\quad \left|\arg(y)\right|<\frac{\pi}{2}\Theta, \qquad \ldots (1.15)$$

$$\Omega=\tau_i\left(\sum_{j=1}^{p_i}\alpha_{ji}-\sum_{j=1}^{q_i}\beta_{ji}\right)+\tau_{i'}\left(\sum_{j=1}^{p_{i''}}C_{ji'}-\sum_{j=1}^{q_{i''}}D_{ji'}\right)<0 \quad , \qquad \ldots (1.16)$$

$$\Delta=\tau_i\left(\sum_{j=1}^{p_i}A_{ji}-\sum_{j=1}^{q_i}B_{ji}\right)+\tau_{i'}\left(\sum_{j=1}^{p_{i''}}E_{ji'}-\sum_{j=1}^{q_{i''}}F_{ji'}\right)<0, \qquad \ldots (1.17)$$

where

$$\Xi=\tau_i\left(\sum_{j=n+1}^{p_i}\alpha_{ji}-\sum_{j=1}^{q_i}\beta_{ji}\right)+\tau_{i'}\left(\sum_{j=m_1+1}^{q_{i'}}D_{ji'}-\sum_{j=n+1}^{p_{i'}}C_{ji'}\right)+\sum_{j=1}^{n_1}C_j>0, \qquad \ldots (1.18)$$

$$\Theta=\tau_i\left(-\sum_{j=n+1}^{p_i}A_{ji}-\sum_{j=1}^{q_i}B_{ji}\right)-\tau_{i''}\left(\sum_{j=n_2+1}^{p_{i''}}E_{ji''}+\sum_{j=m_2+1}^{q_{i''}}F_{ji''}\right)+\sum_{j=1}^{n_2}E_j-\sum_{j=1}^{m_2}F_j>0. \qquad \ldots (1.19)$$

It may be noted that as Aleph function of two variables defined by (1.13) in terms of double Barnes integral is most general in nature, which includes a number of special functions which can be deduced by assigning suitable values to the parameters.

Remark 1. For $\tau_i=\tau_{i'}=\tau_{i''}=1, \left(i=\overline{1,r},\ i'=\overline{1,r'},\ i''=\overline{1,r''}\right)$, Aleph-function of two variables (1.13) reduces to the I -function of two variables due to Sharma *et al.*[13] (also, see [14]).

Remark 2. Further, if we set $r=r'=r''=1$, then (1.13) yields the H - function of two variables given by Mittal and Gupta[15] (also, see [16]).

For the details of Aleph-function of one variable, its fractional calculus formulas and applications the reader can refer the work [3, 4, 7, 17-19].

2. GENERALIZED FRACTIONAL INTEGRALS OF THE ALEPH-FUNCTION OF TWO-VARIABLES

In this section we will establish two generalized fractional integration formulas for Aleph-function of two variables (1.13). The conditions as given in (1.15)-(1.18) hold true.

Theorem 1. Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\rho\in\mathbb{C}$, $\mathrm{Re}(\gamma)>0,\mu>0,\ \upsilon>0$, and

$$\mathrm{Re}(\rho)+\mu\min_{1\le j\le m_1}\mathrm{Re}\left(\frac{d_j}{D_j}\right)+\upsilon\min_{1\le j\le m_2}\mathrm{Re}\left(\frac{f_j}{F_j}\right)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right].$$

Further, let the constants $\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C};A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+$ with $(i=1,...,r;\ j=1,...,p_i)$; $c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+$ $(i'=1,...,r';\ j=1,...,p_{i'})$;

$e_j,f_j,e_{ji''},f_{ji''}\in\mathbb{C};\ E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,...,r'';\ j=1,...,p_{i''})$, also satisfy the conditions are given by (1.15)-(1.19). Then, the left-sided generalized fractional integral of the Aleph-function of two variables exists and the following relation holds:

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{\mu};\sigma t^{\upsilon}\right]\right)\right\}(x)=\ x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times\ \aleph^{0,n+3:m_1,n_1;m_2,n_2}_{p_i+3,q_i+3,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{matrix}\lambda x^{\mu}\\ \sigma x^{\upsilon}\end{matrix}\left|\begin{matrix}A,(1-\rho;\mu,\upsilon),(1-\rho+\alpha+\alpha'+\beta-\gamma;\mu,\upsilon),\\ B,(1-\rho+\alpha+\alpha'-\gamma;\mu,\upsilon),(1-\rho-\beta';\mu,\upsilon),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho+\alpha'-\beta';\mu,\upsilon):C;E\\ (1-\rho+\alpha'+\beta-\gamma;\mu,\upsilon):D;F\end{matrix}\right],\ldots\ (2.1)$$

here, we let $A=\left(a_j;\alpha_j,A_j\right)_{1,n},\left[\tau_j\left(a_{ji};\alpha_{ji},A_{ji}\right)\right]_{n+1,p_i}$; $B=\left[\tau_j\left(b_{ji};\beta_{ji},B_{ji}\right)\right]_{1,q_i}$; $C=\left(c_j,C_j\right)_{1,n_1},\left[\tau_j\left(c_{ji'},C_{ji'}\right)\right]_{n_1+1,p_{i'}}$;

$D=\left(d_j,D_j\right)_{1,m_1},\left[\tau_j\left(d_{ji'},D_{ji'}\right)\right]_{m_2+1,q_{i'}}$; $E=\left(e_j,E_j\right)_{1,n_2},\left[\tau_j\left(e_{ji''},E_{ji''}\right)\right]_{n_2+1,p_{i''}}$; $F=\left(f_j,F_j\right)_{1,m_2},\left[\tau_j\left(f_{ji''},F_{ji''}\right)\right]_{m_2+1,q_{i''}}$.

Proof: In order to prove (2.1), we first express Aleph-function of two variables occurring on the left hand side of (2.1) in terms of Mellin-Barnes contour integral with the help of (1.13), and interchanging the order of integration, we have the following form:

$$I=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\theta_1(s)\theta_2(\xi)\ \lambda^{-s}\sigma^{-\xi}\,ds\,d\xi\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\rho-\mu s-\upsilon\xi-1}\right)(x)$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\theta_1(s)\theta_2(\xi)\ \lambda^{-s}\sigma^{-\xi}\ \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\rho-\mu s-\upsilon\xi-1}\right)(x)\,ds\,d\xi\ .$$

By using power function formula (1.11), we arrive at

$$I = \frac{1}{(2\pi\omega)^2} \int_{L_1}\int_{L_2} \frac{\Gamma(\rho-\mu s-\upsilon\xi)\,\Gamma(\rho-\alpha-\alpha'-\beta+\gamma-\mu s-\upsilon\xi)\,\Gamma(\rho-\alpha'+\beta'-\mu s-\upsilon\xi)}{\Gamma(\rho-\alpha-\alpha'+\gamma-\mu s-\upsilon\xi)\,\Gamma(\rho-\alpha'-\beta+\gamma-\mu s-\upsilon\xi)\,\Gamma(\rho+\beta'-\mu s-\upsilon\xi)}$$

$$\times\, \phi(s,\xi)\,\theta_1(s)\,\theta_2(\xi)\; x^{\rho-\mu s-\upsilon\xi-\alpha-\alpha'+\gamma-1}\,\lambda^{-s}\sigma^{-\xi} ds\, d\xi .$$

Finally, re-interpreting the Mellin-Barnes counter integral in terms of the Aleph-function of two variables, we get the expression as in the right-hand side of (2.1). In view of the relation (1.3), then we get the following corollary concerning Saigo fractional integral operator [11]:

Corollary 1.1. Let $\alpha,\beta,\gamma,\rho,\lambda,\sigma\in\mathbb{C}, \mathrm{Re}(\alpha)>0,\ \mu>0, \upsilon>0$ and let the constraints

$$\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C}; A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+\ (i=1,...,r;\ j=1,...,p_i);$$
$$c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,...,r';\ j=1,...,p_{i'});\ e_j,f_j,e_{ji''},f_{ji''}\in\mathbb{C};$$

$E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,...,r'';\ j=1,...,p_{i''})$, and also satisfy the condition

$\mathrm{Re}(\rho)+\mu\min\limits_{1\le j\le m_1}\mathrm{Re}\left(\frac{d_j}{D_j}\right)+\upsilon\min\limits_{1\le j\le m2}\mathrm{Re}\left(\frac{f_j}{F_j}\right)>\max\left[0,\mathrm{Re}(\beta-\gamma)\right]$. Then, the left-sided Saigo fractional integral of the Aleph-function of two variables exist and the following relation holds true:

$$\left\{I_{0+}^{\alpha,\beta,\gamma}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{\mu};\sigma t^{\upsilon}\right]\right)\right\}(x)=\ x^{\rho-\beta-1}$$

$$\times\ \aleph^{0,n+2:m_1,n_1;m_2,n_2}_{p_i+2,q_i+2,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{matrix}\lambda x^{\mu}\\ \sigma x^{\upsilon}\end{matrix}\left|\begin{matrix}A,(1-\rho;\mu,\upsilon),(1-\rho+\beta-\gamma;\mu,\upsilon):C;E\\ B,(1-\rho+\beta;\mu,\upsilon),(1-\rho-\alpha-\gamma;\mu,\upsilon):D;F\end{matrix}\right.\right], \qquad \dots (2.2)$$

where, A,B,C,D,E and F are the same as given in Theorem 1.For $\beta=-\alpha$ in (2.2), the Saigo integral operator reduces to Riemann-Liouville integral operator [20] and we obtain the following result:

Corollary 1.2. Let $\alpha,\rho,\lambda,\sigma\in\mathbb{C}, \mathrm{Re}(\alpha)>0,\ \mu>0, \upsilon>0$; then, the left-sided Riemann-Liouville fractional integral of the Aleph-function of two variables exist and the following relation holds:

$$\left\{I_{0+}^{\alpha}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{\mu};\sigma t^{\upsilon}\right]\right)\right\}(x)=\ x^{\rho+\alpha-1}$$
$$\times\ \aleph^{0,n+1:m_1,n_1;m_2,n_2}_{p_i+1,q_i+1,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{matrix}\lambda x^{\mu}\\ \sigma x^{\upsilon}\end{matrix}\left|\begin{matrix}A,(1-\rho;\mu,\upsilon):C;E\\ B,(1-\rho-\alpha;\mu,\upsilon):D;F\end{matrix}\right.\right] \dots . (2.3)$$

Further, if we set $\beta=0$ in (2.2), then we can easily obtain result concerning left-sided Erdélyi-Kober fractional integral operator.

Theorem 2. Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\rho\in\mathbb{C}$, $\mathrm{Re}(\gamma)>0, \mu>0, \upsilon>0$, and $\mathrm{Re}(\rho)-$

$$\mu \min_{1\le j\le m_1} \mathrm{Re}\left(\frac{d_j}{D_j}\right)-\upsilon \min_{1\le j\le m_2} \mathrm{Re}\left(\frac{f_j}{F_j}\right)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right].$$

Further, let the constants $\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C}; A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+$ with $(i=1,...,r;\ j=1,...,p_i);\ c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,...,r';\ j=1,...,p_{i'});$

$e_j, f_j, e_{ji''}, f_{ji''}\in\mathbb{C};\ E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,...,r'';\ j=1,...,p_{i''})$, and also satisfy the conditions are given by (1.15) - (1.19). Then, the right-sided generalized fractional integral of the Aleph-function of two variables exists and the following relation holds:

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\ \aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{-\mu};\sigma t^{-\upsilon}\right]\right)\right\}(x)=\ x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times\ \aleph^{0,n+3:m_1,n_1;m_2,n_2}_{p_i+3,q_i+3,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{matrix}\lambda x^{-\mu}\\ \sigma x^{-\upsilon}\end{matrix}\left|\begin{matrix}A,(\rho-\alpha-\alpha'+\gamma;\mu,\upsilon),(\rho-\alpha-\beta'+\gamma;\mu,\upsilon),\\ B,(\rho;\mu,\upsilon),(\rho-\alpha-\alpha'-\beta'+\gamma;\mu,\upsilon),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(\rho+\beta;\mu,\upsilon):C;E\\ (\rho-\alpha+\beta;\mu,\upsilon):D;F\end{matrix}\right],\ldots\ (2.4)$$

where, A,B,C,D,E and F are the same as given in **Theorem 1**.

Proof: The proof of result asserted by **Theorem 2** runs parallel to that of **Theorem 1**. Here we use (1.12) instead of (1.11). The details are, therefore, being omitted.

If we follow **Theorem 2** in respective case $\alpha'=\beta'=0,\beta=-\gamma,\alpha=\alpha+\beta,\gamma=\alpha$. Then, we arrive at the following corollary concerning right-sided Saigo fractional integral operator:

Corollary 2.1. Let $\alpha,\beta,\gamma,\rho,\lambda,\sigma\in\mathbb{C},\mathrm{Re}(\alpha)>0,\ \mu>0,\upsilon>0$ and let the constraints

$\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C}; A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+\ (i=1,...,r;\ j=1,...,p_i)$; $c_j,d_j,d_{ji'}$, $c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,...,r';\ j=1,...,p_{i'});\ e_j, f_j, e_{ji''}, f_{ji''}\in\mathbb{C};\ E_j,F_j,E_{ji''},F_{ji''}$

$\in\mathbb{R}_+$ with $(i''=1,...,r'';\ j=1,...,p_{i''})$, and also satisfy the condition

$$\mathrm{Re}(\rho)-\mu \min_{1\le j\le m_1} \mathrm{Re}\left(\frac{d_j}{D_j}\right)-\upsilon \min_{1\le j\le m2} \mathrm{Re}\left(\frac{f_j}{F_j}\right)<1+\min\left[\mathrm{Re}(\beta),\mathrm{Re}(\gamma)\right].$$

Then, the right-sided Saigo fractional integral of the Aleph-function of two variables exist and we get following relation:

$$\left\{I_{-}^{\alpha,\beta,\gamma}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{-\mu};\sigma t^{-\upsilon}\right]\right)\right\}(x)=x^{\rho-\beta-1}$$

$$\times\ \aleph^{0,n+2:m_1,n_1;m_2,n_2}_{p_i+2,q_i+2,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|c}\lambda x^{-\mu} & A,(\rho-\beta;\mu,\upsilon),(\rho-\gamma;\mu,\upsilon):C;E\\ \sigma x^{-\upsilon} & B,(\rho;\mu,\upsilon),(\rho-\alpha-\beta-\gamma;\mu,\upsilon):D;F\end{array}\right], \qquad \ldots (2.5)$$

where, A,B,C,D,E and F are the same as given in **Theorem 1**.For $\beta=-\alpha$ in (2.5), then, we obtain the following corollary:

Corollary 2.2. Let $\alpha,\rho,\lambda,\sigma\in\mathbb{C},\mathrm{Re}(\alpha)>0,\ \mu>0,\upsilon>0$; then, the right-sided Riemann-Liouville fractional integral of the Aleph-function of two variables exist and the following relation holds true:

$$\left\{I_{-}^{\alpha}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{-\mu};\sigma t^{-\upsilon}\right]\right)\right\}(x)=x^{\rho+\alpha-1}$$

$$\times\ \aleph^{0,n+1:m_1,n_1;m_2,n_2}_{p_i+1,q_i+1,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|c}\lambda x^{-\mu} & A,(\rho+\alpha;\mu,\upsilon):C;E\\ \sigma x^{-\upsilon} & B,(\rho;\mu,\upsilon):D;F\end{array}\right]. \qquad \ldots (2.6)$$

Moreover, if we set $\beta=0$ in (2.5), then we can obtain result concerning right-sided Erdélyi-Kober fractional integral operator.

3. GENERALIZED FRACTIONAL DERIVATIVES OF THE ALEPH-FUNCTION OF TWO-VARIABLES

In this section we will establish two generalized fractional derivative formulas for Aleph-function of two variables. The conditions as given in (1.15)-(1.18) hold true also.

Theorem 3. Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\rho\in\mathbb{C},\ \mathrm{Re}(\gamma)>0,(\mu,\upsilon>0)$, and

$$\mu\max_{1\le j\le m_1}\left[\frac{-\mathrm{Re}(d_j)}{D_j}\right]+\upsilon\max_{1\le j\le m_2}\left[\frac{-\mathrm{Re}(f_j)}{F_j}\right]<\mathrm{Re}(\rho)+\min\left[0,\mathrm{Re}(\beta-\alpha),\mathrm{Re}(\gamma-\alpha-\alpha'-\beta)\right].$$

Further, let the constants $\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C};A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+$ with $(i=1,\ldots,r;\ j=1,\ldots,p_i);\ c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,\ldots,r';\ j=1,\ldots,p_{i'});$

$e_j,f_j,e_{ji''},f_{ji''}\in\mathbb{C};\ E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,\ldots,r'';\ j=1,\ldots,p_{i''})$, and also satisfy the conditions are given by (1.15)-(1.19). Then, the left-sided generalized fractional derivative of the Aleph-function of two variables exists and given by

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{\mu};\sigma t^{\upsilon}\right]\right)\right\}(x)=x^{\rho+\alpha+\alpha'-\gamma-1}$$

$$\times\ \aleph^{0,n+3:m_1,n_1;m_2,n_2}_{p_i+3,q_i+3,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|c}\lambda x^{\mu} & A,(1-\rho;\mu,\upsilon),(1-\rho-\alpha-\alpha'-\beta'+\gamma;\mu,\upsilon),\\ \sigma x^{\upsilon} & B,(1-\rho-\alpha-\alpha'+\gamma;\mu,\upsilon),(1-\rho+\beta;\mu,\upsilon),\end{array}\right.$$

$$\left.\begin{matrix}(1-\rho-\alpha+\beta;\mu,\upsilon):C;E\\(1-\rho-\alpha-\beta'+\gamma;\mu,\upsilon):D;F\end{matrix}\right],\ldots \quad (3.1)$$

here, A,B,C,D,E and F are the same as given in **Theorem 1**.

Proof: By expressing the Aleph-function of two variables occurring on the left hand side of (3.1) (denoted by $\mathcal{D}$) in terms of Mellin-Barnes contour integral with the help of (1.13), and interchanging the order of differentiation, we have

$$\mathcal{D}=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\theta_1(s)\theta_2(\xi)\,\lambda^{-s}\sigma^{-\xi}\,ds\,d\xi\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\rho-\mu s-\upsilon\xi-1}\right)(x)$$

$$=\frac{d^n}{dx^n}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\theta_1(s)\theta_2(\xi)\,\lambda^{-s}\sigma^{-\xi}\,\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}\,t^{\rho-\mu s-\upsilon\xi-1}\right)(x)ds\,d\xi\ ,$$

where $n=\left[\operatorname{Re}(\gamma)+1\right]$. Now, by applying (1.11), we have

$$\mathcal{D}=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\frac{\Gamma(\rho-\mu s-\upsilon\xi)\,\Gamma(\rho+\alpha+\alpha'+\beta'-\gamma-\mu s-\upsilon\xi)\,\Gamma(\rho+\alpha-\beta-\mu s-\upsilon\xi)}{\Gamma(\rho+\alpha+\alpha'-\gamma+n-\mu s-\upsilon\xi)\,\Gamma(\rho+\alpha+\beta'-\gamma-\mu s-\upsilon\xi)\,\Gamma(\rho-\beta-\mu s-\upsilon\xi)}$$

$$\times\,\phi(s,\xi)\theta_1(s)\theta_2(\xi)\left(\frac{d^n}{dx^n}x^{\rho-\mu s-\upsilon\xi+\alpha+\alpha'-\gamma+n-1}\right)\lambda^{-s}\sigma^{-\xi}ds\,d\xi\,.$$

By re-interpreting the Mellin-Barnes counter integral in terms of the Aleph-function of two variables, and using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\,x^{m-n}\quad(m\geq n)$, we find the desired result (3.1). This completes the proof.

In view of the relation (1.9), then we arrive at the following corollary concerning Saigo fractional derivative operator [11]:

Corollary 3.1. Let $\alpha,\beta,\gamma,\rho,\lambda,\sigma\in\mathbb{C},\operatorname{Re}(\alpha)>0,\ \mu>0,\upsilon>0$ and let the constraints

$$\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C};A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+\ (i=1,\ldots,r;\,j=1,\ldots,p_i);$$
$$c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,\ldots,r';\,j=1,\ldots,p_{i'});\ e_j,f_j,e_{ji''},f_{ji''}\in\mathbb{C};$$

$E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,\ldots,r'';\,j=1,\ldots,p_{i''})$, also satisfy the condition

$\mu\max\limits_{1\leq j\leq m_1}\left[\frac{-\operatorname{Re}(d_j)}{D_j}\right]+\upsilon\max\limits_{1\leq j\leq m_2}\left[\frac{-\operatorname{Re}(f_j)}{F_j}\right]<\operatorname{Re}(\rho)+\max\left[0,\operatorname{Re}(\alpha+\beta+\gamma)\right]$. Then, the left -sided Saigo fractional derivative of the Aleph-function of two variables exist and we have the following:

$$\left\{D_{0+}^{\alpha,\beta,\gamma}\left(t^{\rho-1}\,\aleph_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}^{0,n:m_1,n_1;m_2,n_2}\left[\lambda t^{\mu};\sigma t^{\upsilon}\right]\right)\right\}(x)=\,x^{\rho+\beta-1}$$

$$\times \aleph^{0,n+2:m_1,n_1;m_2,n_2}_{p_i+2,q_i+2,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|c}\lambda x^{\mu} & A,(1-\rho;\mu,\upsilon),(1-\rho-\alpha-\beta-\gamma;\mu,\upsilon):C;E \\ \sigma x^{\upsilon} & B,(1-\rho-\beta;\mu,\upsilon),(1-\rho-\gamma;\mu,\upsilon):D;F\end{array}\right], \quad \ldots (3.2)$$

where, A,B,C,D,E and F are the same as given in **Theorem 1**.For $\beta=-\alpha$ in (3.2), we have the following result concerning left-sided Riemann-Liouville fractional derivative:

Corollary 3.2. Let $\alpha,\rho,\lambda,\sigma\in\mathbb{C},\mathrm{Re}(\alpha)>0,\ \mu>0,\upsilon>0$; then, the left-sided Riemann-Liouville fractional integral of the Aleph-function of two variables exist and the following relation holds true:

$$\left\{D^{\alpha}_{0+}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{\mu};\sigma t^{\upsilon}\right]\right)\right\}(x)=\ x^{\rho-\alpha-1}$$

$$\times\ \aleph^{0,n+1:m_1,n_1;m_2,n_2}_{p_i+1,q_i+1,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|c}\lambda x^{\mu} & A,(1-\rho;\mu,\upsilon):C;E \\ \sigma x^{\upsilon} & B,(1-\rho+\alpha;\mu,\upsilon):D;F\end{array}\right]\ldots.\ (3.3)$$

Next, if we set $\beta=0$ in (3.2), then we can easily obtain result concerning left-sided Erdélyi-Kober fractional derivative operator.

Theorem 4. Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\lambda,\rho\in\mathbb{C},\ \mathrm{Re}(\gamma)>0,\mu>0,\ \upsilon>0$, and

$$1+\mu\min_{1\le j\le m_1}\left[\frac{1-\mathrm{Re}(c_j)}{C_j}\right]+\upsilon\min_{1\le j\le m_2}\left[\frac{1-\mathrm{Re}(e_j)}{E_j}\right]>\mathrm{Re}(\rho)-\min\{0,\mathrm{Re}(\gamma-\alpha-\alpha'-n),\mathrm{Re}(-\alpha'-\beta+\gamma),-\mathrm{Re}(\beta')\}.$$

Also, let the constants $\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C};A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+$ with $(i=1,\ldots,r;\ j=1,\ldots,p_i);\ c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,\ldots,r';\ j=1,\ldots,p_{i'});$

$e_j,f_j,e_{ji''},f_{ji''}\in\mathbb{C};\ E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,\ldots,r'';\ j=1,\ldots,p_{i''})$, and satisfy the conditions are given by (1.15)-(1.19). Then, the right-sided generalized fractional derivative of the Aleph-function of two variables exists and the following relation holds true:

$$\left\{D^{\alpha,\alpha',\beta,\beta',\gamma}_{-}\left(t^{\rho-1}\,\aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{-\mu};\sigma t^{-\upsilon}\right]\right)\right\}(x)=\ x^{\rho+\alpha+\alpha'-\gamma-1}$$

$$\times\ \aleph^{0,n+3:m_1,n_1;m_2,n_2}_{p_i+3,q_i+3,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{array}{c|c}\lambda x^{-\mu} & A,(\rho+\alpha+\alpha'-\gamma;\mu,\upsilon),(\rho+\alpha'+\beta-\gamma;\mu,\upsilon), \\ \sigma x^{-\upsilon} & B,(\rho;\mu,\upsilon),(\rho+\alpha+\alpha'+\beta-\gamma;\mu,\upsilon),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho-\beta';\mu,\upsilon):C;E \\ (\rho+\alpha'-\beta';\mu,\upsilon):D;F\end{array}\right],\ldots\ (3.4)$$

where, A,B,C,D,E and F are the same as given in **Theorem 1**.

Proof: A similar argument as in the proof of Theorem 3 will establish the result (3.4) under the given conditions. So the details of the proof are omitted. Setting $\alpha'=\beta'=0,\beta=-\gamma,\alpha=\alpha+\beta,\gamma=\alpha$ in

Theorem 4 yields an identity regarding the right-sided Saigo fractional derivative operator [11] asserted by the following corollary:

Corollary 4.1. Let $\alpha,\beta,\gamma,\rho,\lambda,\sigma\in\mathbb{C},\mathrm{Re}(\alpha)>0,\ \mu>0,\upsilon>0$ and let the constraints

$$\tau_i,\tau_{i'},\tau_{i''}\in\mathbb{R}_+;\ a_j,a_{ji},b_{ji}\in\mathbb{C};A_j,\alpha_j,A_{ji},B_{ji},\alpha_{ji},\beta_{ji}\in\mathbb{R}_+\ (i=1,\ldots,r;\ j=1,\ldots,p_i)$$
$$c_j,d_j,d_{ji'},c_{ji'}\in\mathbb{C};\ C_j,D_j,C_{ji'},D_{ji'}\in\mathbb{R}_+\ (i'=1,\ldots,r';\ j=1,\ldots,p_{i'});\ e_j,f_j,e_{ji''},f_{ji''}\in\mathbb{C};$$

$E_j,F_j,E_{ji''},F_{ji''}\in\mathbb{R}_+$ with $(i''=1,\ldots,r'';\ j=1,\ldots,p_{i''})$, also satisfy the condition

$$1+\mu\min_{1\le j\le m_1}\left[\frac{1-\mathrm{Re}(c_j)}{C_j}\right]+\upsilon\min_{1\le j\le m_2}\left[\frac{1-\mathrm{Re}(e_j)}{E_j}\right]>\mathrm{Re}(\rho)+\max\left\{\mathrm{Re}(\beta)+\left[\mathrm{Re}(\beta)\right]+1,-\mathrm{Re}(\alpha+\gamma)\right\}.$$

Then, the right-sided Saigo fractional derivative of the Aleph-function of two variables exists and is given as follows:

$$\left\{D_{-}^{\alpha,\beta,\gamma}\left(t^{\rho-1}\ \aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{-\mu};\sigma t^{-\upsilon}\right]\right)\right\}(x)=\ x^{\rho+\beta-1}$$

$$\times\ \aleph^{0,n+2:m_1,n_1;m_2,n_2}_{p_i+2,q_i+2,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{matrix}\lambda x^{-\mu}\\ \sigma x^{-\upsilon}\end{matrix}\middle|\begin{matrix}A,(\rho+\beta;\mu,\upsilon),(\rho-\alpha-\gamma;\mu,\upsilon):C;E\\ B,(\rho;\mu,\upsilon),(\rho+\beta-\gamma;\mu,\upsilon):D;F\end{matrix}\right],\qquad\ldots(3.5)$$

where, A,B,C,D,E and F are the same as given in **Theorem 1**.For $\beta=-\alpha$ in (3.5), we obtain the following corollary involving Riemann-Liouville fractional derivative operators:

Corollary 4.2. Let $\alpha,\rho,\lambda,\sigma\in\mathbb{C},\mathrm{Re}(\alpha)>0,\ \mu>0,\upsilon>0$; then, the right-sided Riemann-Liouville fractional integral of the Aleph-function of two variables exist and the following relation holds true:

$$\left\{D_{-}^{\alpha}\left(t^{\rho-1}\ \aleph^{0,n:m_1,n_1;m_2,n_2}_{p_i,q_i,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\lambda t^{-\mu};\sigma t^{-\upsilon}\right]\right)\right\}(x)=\ x^{\rho-\alpha-1}$$

$$\times\ \aleph^{0,n+1:m_1,n_1;m_2,n_2}_{p_i+1,q_i+1,\tau_i,r:p_{i'},q_{i'},\tau_{i'},r';p_{i''},q_{i''},\tau_{i''},r''}\left[\begin{matrix}\lambda x^{-\mu}\\ \sigma x^{-\upsilon}\end{matrix}\middle|\begin{matrix}A,(\rho-\alpha;\mu,\upsilon):C;E\\ B,(\rho;\mu,\upsilon):D;F\end{matrix}\right]\ldots.\ (3.6)$$

Further, certain result involving Erdélyi-Kober fractional derivative operator can also be obtained by setting $\beta=0$ in Corollary 4.1. We omit details here.

Remark 3: For more details of generalized fractional calculus operators involving special functions and their application the reader can refer the recent published work [21-25].

4. CONCLUDING REMARKS AND SPECIAL CASES

This section deals with certain special cases of the **Theorem 1**. Since the Aleph-function of two variables is very general, it contains, as its special cases, many special functions. Setting $\tau_i=\tau_{i'}=\tau_{i''}=1\ \left(i\in\overline{1,r};\ i'\in\overline{1,r'};\ i''\in\overline{1,r''}\right)$ in (1.13) yields the I-function of two variables (see, Sharma and Mishra [13]) whose further special case when $r=r'=r''=1$ reduces to the H-function of two variables.

Here, among numerous special cases of the four main results, only a few cases of main result (2.1) are demonstrated as in the following results.

(i.). If we put $\tau_i = \tau_{i'} = \tau_{i''} = 1 \left(i \in \overline{1,r};\ i' \in \overline{1,r'};\ i'' \in \overline{1,r''} \right)$ in (2.1), then we have the following result in the term of I-function of two variables [13]:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\rho-1} \ I_{p_i,q_i,r:p_{i'},q_{i'},r';p_{i''},q_{i''},r''}^{0,n:m_1,n_1;m_2,n_2} \left[\lambda t^{\mu}; \sigma t^{\upsilon} \right] \right) \right\}(x) = x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times\ I_{p_i+3,q_i+3,r:p_{i'},q_{i'},r';p_{i''},q_{i''},r''}^{0,n+3:m_1,n_1;m_2,n_2} \left[\begin{array}{c|l} \lambda x^{\mu} & L,(1-\rho;\mu,\upsilon),(1-\rho+\alpha+\alpha'+\beta-\gamma;\mu,\upsilon), \\ \sigma x^{\upsilon} & M,(1-\rho+\alpha+\alpha'-\gamma;\mu,\upsilon),(1-\rho-\beta';\mu,\upsilon), \end{array} \right.$$

$$\left. \begin{array}{c} (1-\rho+\alpha'-\beta';\mu,\upsilon):N;V \\ (1-\rho+\alpha'+\beta-\gamma;\mu,\upsilon):U;W \end{array} \right], \ldots \text{ (4.1) here,}$$

$$L = \left(a_j;\alpha_j,A_j\right)_{1,n}, \left(a_{ji};\alpha_{ji},A_{ji}\right)_{n+1,p_i}; M = \left(b_j;\beta_j,B_j\right)_{1,m}, \left(b_{ji};\beta_{ji},B_{ji}\right)_{m+1,q_i}; N = \left(c_j,C_j\right)_{1,n_1}, \left(c_{ji'},C_{ji'}\right)_{n_1+1,p_{i'}};$$

$$U = \left(d_j,D_j\right)_{1,m_1}, \left(d_{ji'},D_{ji'}\right)_{m_2+1,q_{i'}}; V = \left(e_j,E_j\right)_{1,n_2}, \left(e_{ji''},E_{ji''}\right)_{n_2+1,p_{i''}}; W = \left(f_j,F_j\right)_{1,m_2}, \left(f_{ji''},F_{ji''}\right)_{m_2+1,q_{i''}}.$$

Further, if we set $r = r' = r'' = 1$ in (4.2), then we arrive at the following result in the term of H-function of two variables [16, 26]:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\rho-1} \ H_{p,q:p_1,q_1;p_2,q_2}^{0,n:m_1,n_1;m_2,n_2} \left[\lambda t^{\mu}; \sigma t^{\upsilon} \right] \right) \right\}(x) = x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times\ H_{p+3,q+3:p_1,q_1;p_2,q_2}^{0,n+3:m_1,n_1;m_2,n_2} \left[\begin{array}{c|l} \lambda x^{\mu} & P,(1-\rho;\mu,\upsilon),(1-\rho+\alpha+\alpha'+\beta-\gamma;\mu,\upsilon), \\ \sigma x^{\upsilon} & Q,(1-\rho+\alpha+\alpha'-\gamma;\mu,\upsilon),(1-\rho-\beta';\mu,\upsilon), \end{array} \right.$$

$$\left. \begin{array}{c} (1-\rho+\alpha'-\beta';\mu,\upsilon):R;T \\ (1-\rho+\alpha'+\beta-\gamma;\mu,\upsilon):S;Z \end{array} \right], \ldots \text{ (4.2)}$$

here, $P = \left(a_j;\alpha_j,A_j\right)_{1,p}; Q = \left(b_j;\beta_j,B_j\right)_{1,q}; R = \left(c_j,C_j\right)_{1,p_1}; S = \left(d_j,D_j\right)_{1,q_1}; T = \left(e_j,E_j\right)_{1,p_2}; Z = \left(f_j,F_j\right)_{1,q_2}.$

(ii). Next, if we put $n = 0 = p_i = q_i \left(i \in \overline{1,r} \right)$ in Theorem 1, then we obtain the product of two Aleph-functions, and we have the following known result given by Saxena *et al.* [2, p. 636, eq. (3.1)]:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\rho-1} \ \aleph_{p_i,q_i,\tau_i,r}^{m_1,n_1} \left[\lambda t^{\mu} \right] \ \aleph_{p_{i'},q_{i'},\tau_{i'},r'}^{m_2,n_2} \left[\sigma t^{\upsilon} \right] \right) \right\}(x) = x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times\ \aleph_{3,3:p_i,q_i,\tau_i,r;p_{i'},q_{i'},\tau_{i'},r'}^{0,3:m_1,n_1;m_2,n_2} \left[\begin{array}{c|l} \lambda x^{\mu} & (1-\rho;\mu,\upsilon),(1-\rho+\alpha+\alpha'+\beta-\gamma;\mu,\upsilon), \\ \sigma x^{\upsilon} & (1-\rho+\alpha+\alpha'-\gamma;\mu,\upsilon),(1-\rho-\beta';\mu,\upsilon), \end{array} \right.$$

$$\left.\begin{matrix}(1-\rho+\alpha'-\beta';\mu,\upsilon):A';C' \\ (1-\rho+\alpha'+\beta-\gamma;\mu,\upsilon):B';D'\end{matrix}\right], \quad \ldots (4.3)$$

here, $A'=\left(a_j,A_j\right)_{1,n_1},\left[\tau_j\left(a_j,A_j\right)\right]_{n_1+1,p_i}$; $B'=\left(b_j,B_j\right)_{1,m_1},\left[\tau_j\left(b_j,B_j\right)\right]_{m_1+1,q_i}$; $C'=\left(c_j,C_j\right)_{1,n_2},\left[\tau_j\left(c_j,C_j\right)\right]_{n_2+1,p_{i'}}$

and $D'=\left(d_j,D_j\right)_{1,m_2},\left[\tau_j\left(d_j,D_j\right)\right]_{m_2+1,q_{i'}}$.

For more detail of product of Aleph-functions and its several applications the reader can refer the work done by Saxena *et al.*[2, 5], Kumar and Choi [6].

Remark 4: We can easily obtain the similar special cases for **Theorem 2-4**. Besides this, on account of the general nature of the generalized fractional calculus operators and H -function of two variables a large number of new and known results involving special functions notably Mittag-Leffler function, Whittaker function and Bessel function of the first kind can be developed as special cases, whose detailed accounts are omitted.

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