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# The Propagation of Spin Coherent States

**SEDDIK Sanaa**

LPHE-Modeling and Simulation, Faculty of Sciences, University Mohammed V, Rabat, Morocco

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**Abstract:** We derive the semi-classical propagation of spin coherent states in complex phase space. We construct a path integral representation for the propagator of such systems and find the appropriate classical trajectories. As special cases we consider two independent systems.

**Keywords:** Spin coherent states, Path integral, Quantum propagator.

## 1. INTRODUCTION

In classical mechanics, the dynamics is directly determined by a set of differential equations on a phase space, and the structures to ensure the integrability are the poisson brackets. In quantum mechanics, although the Schrödinger equation is linear, or more precisely, the wave functions are linear vectors in the Hilbert space, the basic structure to determine quantum dynamics, including the determination of the Hilbert space, are the commutation relations[1-4] . Mathematically these structures (the Poisson brackets in classical mechanics and commutation relations in quantum mechanics) lie at the same level, namely, they serve as the algebraic structure of group theory. Therefore, it is natural to ask whether we can establish the concept of quantum integrability in finite systems on the basis of group structure in such a way that classical integrability can be shown to be a consequence of the general quantum integrability theory[5]. The answer should be positive. In fact, it is well known that the mathematical basis of quantum theory is group representation theory. From group theory, we can completely determine whether a system can be analytically and exactly solved, which can be defined as a criterion of integrability.

In our work, we concentrate in path integral approach, especially the approach which is the well-known Feynman path integral formalism. Here, one expresses quantum mechanics in terms of a classical Lagrangian by a path integral and then for given initial and final states integrates over all possible paths[1,6,7]. This semi-classical behaviour of such systems has drawn attention for quite a long time. One natural representation for the study of this limit is that of coherent states. The semi-classical limit of the coherent state propagator for both the Weyl and the $SU(2)$ group has already been studied in detail. The purpose of this paper is to derive the semi-classical limit of the fermionic coherent states (spin coherent states) propagator for two systems[8].

This paper is organized as follows. Section 2 concerns a semi-classical quantization condition for spin coherent state in general. The coherent state path integral for two independent systems and its semi-classical approximation have been intensively studied in section 3. Concluding remarks close this paper.

## 2. THE QUANTUM PROPAGATOR

The transition matrix elements expressed as a path integrals provide a natural scheme to obtain stationary phase approximations for propagators[1,6,7,9,10]. Elsewhere, in applying the expression for the path integral in terms of an arbitrary continuous representation suitable for analyzing stationary phase approximations. In particular, this expression is formally given by

$$< q', t' \,\middle|\, q, t >= \lim_{\epsilon\to 0} \check{\mathrm{T}} \int \exp\{\, i \int_t^{t'} [i < l \,\middle|\, \dot{l} > + \frac{1}{2}\epsilon < \dot{l} \,\middle|\, |(1 - |l >< l|)| \,\middle|\, \dot{l} > -H(l)dt]\} Dl. \tag{1}$$

We are now looking the probability amplitude for the state $|\eta_1>$ at time 0 of going to state $|\eta_2>$ at time $t$, $K\,(\bar{\eta}_2, \eta_1, t)$. We have

$$K\,(\bar{\eta}_2, \eta_1, t) =< \eta_2 \,\middle|\, U(0,t) \,\middle|\, \eta_1 >. \tag{2}$$

The time id divided in regular intervals

$$\tau_k = k\epsilon, \quad k \in [0, N] \quad N \in \mathbb{N}. \tag{3}$$

We have the following correspondence

$$|\eta_1 >\equiv \tau_0, \quad (4)$$

$$|\eta_2 >= \tau_N = t. \quad (5)$$

Such that the operator $U$ is written[11]

$$U(0,t) = U(\tau_0,\tau_N) = U(\tau_{N-1},\tau_N)U(\tau_{N-2},\tau_{N-1}) \times \ldots \times U(\tau_1,\tau_2)U(\tau_0,\tau_1). \quad (6)$$

Therefore, it can correspond to each moment $\tau_k$ an integration variable $\varepsilon_k$, then

$$\begin{aligned} U(0,t) \; &= \int_{\mathbb{C}} d^2\mu_j |\varepsilon_N >< \varepsilon_N| U(\tau_{N-1},\tau_N) \int_{\mathbb{C}} d^2\mu_j |\varepsilon_{N-1} >< \varepsilon_{N-1}| \times \ldots \times \\ & \int_{\mathbb{C}} d^2\mu_j |\varepsilon_1 >< \varepsilon_1| U(\tau_0,\tau_1) \int_{\mathbb{C}} d^2\mu_j |\varepsilon_0 >< \varepsilon_0|. \end{aligned} \quad (7)$$

This implies the kernel[12]

$$\begin{aligned} K\,(\bar{\eta}_2,\eta_1,t) = \int_{\mathbb{C}^{N+1}} \prod_{k=0}^{N} d^{2\mu_j}(\varepsilon_k) < \eta_2|\varepsilon_N >< \varepsilon_0|\eta_1 > \\ \prod_{k=0}^{N} < \varepsilon_k|U(\tau_{k-1},\tau_k)|\varepsilon_{k-1} >, \end{aligned} \quad (8)$$

with ($\hbar$= 1)

$$U(\tau_{k-1},\tau_k) = U(\tau_{k-\epsilon},\tau_k) \simeq e^{-i\epsilon H(\tau_k)}. \quad (9)$$

The expression (8) is rewritten

$$K\,(\bar{\eta}_2,\eta_1,t) = \int_{\mathbb{C}^{N+1}} \prod_{k=0}^{N} d^{2\mu_j}(\varepsilon_k) < \eta_2|\varepsilon_N >< \varepsilon_0|\eta_1 > \prod_{k=1}^{N} < \varepsilon_k \left| e^{-i\epsilon H(\tau_k)} \right| \varepsilon_{k-1} >. \quad (10)$$

For a spin coherent state $|\eta >$, we have

$$L = j\int_0^t \frac{\dot{\bar{\varepsilon}}(\tau)\varepsilon(\tau) - \bar{\varepsilon}(\tau)\dot{\varepsilon}(\tau)}{1+|\varepsilon(\tau)|^2} d\tau - i\int_0^t \hat{H}(\bar{\varepsilon}(\tau),\varepsilon(\tau);\tau)d\tau, \quad (11)$$

and

$$\Omega = j\ln\left(\frac{\big(1+\bar{\varepsilon}_\tau\varepsilon(\tau)\big)(1+\bar{\varepsilon}(0),\varepsilon_0)}{(1+|\varepsilon_\tau|^2)(1+|\varepsilon_0|^2)}\right), \quad (12)$$

thus

$$K(\bar{\varepsilon}_N,\varepsilon_0,t)=\int_{\varepsilon(0)=\varepsilon_0}^{\bar{\varepsilon}(\tau)=\bar{\varepsilon}_\tau} D\mu_j(\varepsilon)\exp(\Phi). \tag{13}$$

Then

$$\Phi = L+\ \Omega, \tag{14}$$

is an action. Now we add to the classical trajectories $\varepsilon_c(\tau)$, $\bar{\varepsilon}_c(\tau)$ two infinitely small trajectory $\delta\varepsilon(\tau)$ and $\delta\bar{\varepsilon}(\tau)$, such as the boundary conditions are $\delta\varepsilon(0)$ = 0 and $\delta\bar{\varepsilon}(0)$ = 0. The variation of the action is

$$\delta\Phi=\int_0^t\{\left(-i\left(\partial_\varepsilon\widehat{H}\right)_c+\frac{2j\dot{\bar{\varepsilon}}_c}{(1+\bar{\varepsilon}_c\varepsilon_c)^2}\right)\delta\varepsilon-(i\left(\partial_{\bar{\varepsilon}}\widehat{H}\right)_c+\frac{2j\dot{\varepsilon}_c}{(1+\bar{\varepsilon}_c\varepsilon_c)^2})\delta\bar{\varepsilon}\}\,d\tau. \tag{15}$$

If we apply the variational principle to that expression, we cancel the integral in $\delta\varepsilon$ and in $\delta\bar{\varepsilon}$. We obtain two independent equations

$$2j\dot{\varepsilon}=-i(1+\bar{\varepsilon}\varepsilon)^2\partial_{\bar{\varepsilon}}\widehat{H},\qquad \varepsilon(0)=\varepsilon_0, \tag{16}$$

$$2j\dot{\bar{\varepsilon}}=+i(1+\bar{\varepsilon}\varepsilon)^2\partial_{\varepsilon}\widehat{H},\qquad \bar{\varepsilon}(t)=\bar{\varepsilon}_t. \tag{17}$$

Therefore

$$u(\tau)=\frac{1}{1+\bar{\varepsilon}_c\varepsilon_c}\exp(i\int_0^t Bds)\frac{\partial_{\varepsilon_c}(\tau)}{\partial_{\varepsilon_0}}\delta\varepsilon_0+\frac{1}{1+\bar{\varepsilon}_c\varepsilon_c}\exp(i\int_0^t Bds)\frac{\partial_{\varepsilon_c}(\tau)}{\partial_{\bar{\varepsilon}_t}}\delta\bar{\varepsilon}_t, \tag{18}$$

then

$$K(\bar{\varepsilon}_N,\varepsilon_0,t)=\exp\left(\Phi_c+i\int_0^t Bds\right)\left(\frac{(1+|\varepsilon_c(0)|^2)(1+|\varepsilon_c(t)|^2)}{2j}\frac{\partial^2\Phi_c}{\partial\varepsilon_0\partial\bar{\varepsilon}_t}\right)^{\frac{1}{2}}, \tag{19}$$

where we have

$$B=\frac{1}{4j}\{\partial_\varepsilon\left[(1+\bar{\varepsilon}\varepsilon)^2\partial_{\bar{\varepsilon}}\widehat{H}\right]+\partial_{\bar{\varepsilon}}[(1+\bar{\varepsilon}\varepsilon)^2\partial_\varepsilon\widehat{H}]\}_c \tag{20}$$

It can also be written as

$$B=\frac{i}{2}(\partial_\varepsilon\dot{\varepsilon}-\partial_{\bar{\varepsilon}}\dot{\bar{\varepsilon}})_{\dot{c}} \tag{21}$$

## 2.2. QUANTUM PROPAGATOR FOR TWO INDEPENDENT SYSTEMS

The propagator is the essential ingredient for quantum dynamical calculations and it is also fundamental in the study of the quantum-classical connection[6,11-13]. We consider two different systems. For the first system we consider the Hamiltonian

$$H(\eta,\bar{\eta}) =< \eta|\omega J_z|\eta >, \tag{22}$$

Thus

$$J_z|\eta >= \frac{1}{(1+\bar{\eta}\eta)^{2j}} \sum_{k=0}^{2j} (k-j) \sqrt{\binom{2j}{k}} \eta^k |j,k-j>. \tag{23}$$

Then

$$< \eta|J_z|\eta >= \frac{1}{(1+\bar{\eta}\eta)^{2j}} \sum_{k,l=0}^{2j} (k-j) \sqrt{\binom{2j}{k}} \sqrt{\binom{2j}{l}} \eta^k \bar{\eta}^l < j,l-j|j,k-j>. \tag{24}$$

with $< j,l-j|j,k-j >= \delta_{l,k}$

$$< \eta|J_z|\eta >= \frac{1}{(1+\bar{\eta}\eta)^{2j}} \sum_{k=0}^{2j} (k-j) \binom{2j}{k} (\bar{\eta}\eta)^k, \tag{25}$$

where

$$\sum_{k=0}^{2j} (k-j) \binom{2j}{k} (\bar{\eta}\eta)^k = 2j\bar{\eta}\eta \sum_{p=0}^{2j-1} \binom{2j-1}{p} (\bar{\eta}\eta)^p = 2j\bar{\eta}\eta(1+\bar{\eta}\eta)^{2j-1}.$$

Thus

$$< \eta|J_z|\eta >= \frac{2j\bar{\eta}\eta}{1+\bar{\eta}\eta} - j. \tag{26}$$

Thus the Hamiltonian function is

$$H(\eta,\bar{\eta}) = \frac{2j\omega\bar{\eta}\eta}{1+\bar{\eta}\eta} - j\omega. \tag{27}$$

Compute $\partial_z H(\eta,\bar{\eta})$ and $\partial_{\bar{z}} H(\eta,\bar{\eta})$

$$\partial_z H(\eta,\bar{\eta}) = \frac{2j\omega\bar{\eta}}{(1+\bar{\eta}\eta)^2}$$

$$\partial_{\bar{z}} H(\eta,\bar{\eta}) = \frac{2j\omega\eta}{(1+\bar{\eta}\eta)^2}$$

Taking up the classical equations of motion (16) and (17) we then have

$$\dot{\bar{\eta}} = i\omega\bar{\eta} \tag{28}$$

$$\dot{\eta} = -i\omega\eta \tag{29}$$

with the initial conditions, conventional solutions of the movement give

$$\varepsilon_c(t) = \varepsilon_0 \exp(-i\omega t) \tag{30}$$

$$\bar{\varepsilon}_c(t) = \bar{\varepsilon}_r \exp\big(-i\omega(\tau - t)\big). \tag{31}$$

The action of the path

$$\Phi_c = ij\omega\,\tau + 2jln\big(1 + \varepsilon_0\bar{\varepsilon}_r e^{-i\omega\,\tau}\big) - jln(1 + \varepsilon_0\bar{\varepsilon}_0) - j(1 + \varepsilon_r\bar{\varepsilon}_r), \tag{32}$$

with

$$B = \omega. \tag{33}$$

Finally, we find the expression of the propagator[11,12]

$$K(\bar{\varepsilon}_r, \varepsilon_0, \tau) = \exp(\Phi_c). \tag{34}$$

For the second system and in the same way, our quantization, the Feymann path integral is an integral over all coordinates. The coordinates are operators in the Hamiltonian formalism. In the path integral case, the argument of the exponential is the action in units of $\hbar$ and the integral are over environment. Let's take a system interacts with an environment that consists of a collection of simple harmonic oscillator modes. The total Hamiltonian of the whole system can be written as[6, 7, 11, 13].

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I \tag{35}$$

where

$$\hat{H}_S = \hbar\omega_1 J_{1+} J_{1-} + \hbar\omega_2 J_{2+} J_{2-} \tag{36}$$

$$\hat{H}_E = \textstyle\sum_k \hbar\omega_k b_k^\dagger b_k \tag{37}$$

$$\hat{H}_I = \textstyle\sum_{l,k} \hbar(g_{lk} J_{l+} b_k + g_{lk}^* J_{l-} b_k^\dagger) \tag{38}$$

are, respectively, the Hamiltonian of the system, the $k-th$ independent environment, and the interactions between them. The operators $J_{l+}$, $J_{l-}$ are corresponding creation and annihilation operator of the $l-th$ mode, with frequency $\omega_l$. The environments is modeled by a set of harmonic oscillators with the annihilation and creation operators $b_k$ and $b_k^\dagger$ (*k=1, 2, ….*), $g_{lk}$ are the coupling constants between the system and the environment.

To apply the influence functional method to an open quantum system, the first step towards the dynamics of the reduced system is to compute the forward and backward propagators between certain initial and final states of the full system by choosing a convenient representation[14-19]. In the present work we use the coherent state representation, in which the basis of the Hilbert space for the environment consists of multi-mode bosonic coherent states

$$|z>=\prod_k|z_k>, \qquad |z_k>=\exp(z_k b_k^{\dagger})|0_k>, \tag{39}$$

and that for the system is the two single-mode fermionic coherent states[8,10]

$$|\eta>=\prod_{l=1}^{2}|\eta_l>, \qquad |\eta_l>=(1+|\eta|^2)^{-j}\exp(\eta J_+)|j,-j>, \tag{40}$$

where $z$ a complex number, denotes the bosonic coherent state and $\eta$ a Grassmaniann number, denotes the fermionic coherent state. The coherent state defined for environment are eigenstates of annihilation operators and are not normalized

$$b_k|z_k>=z_k|z_k>, \tag{41}$$

and the coherent state defined for the system are not eigenstates of annihilation operators and are normalized

$$J_{l-}|\eta_l>\neq\eta_l|\eta_l>. \tag{42}$$

As these coherent states are over-complete, they obey the resolution of identity,

$$\int d\mu(z)|z><z|=I, \qquad \int d\mu(\eta)|\eta><\eta|=I. \tag{43}$$

where the integration measures are defined by

$$d\mu(z)=\prod_k e^{-z_k^* z_k}\frac{dz_k^* dz_k}{2\pi i},$$

and

$$d\mu(\eta)=\prod_l\frac{2j+1}{(1+|\eta|^2)^2}\frac{d\eta_k^* d\eta_k}{2\pi i}.$$

The use of the coherent state representation makes the evaluation of path integrals extremely simple. In the coherent state representation, the classical Hamiltonian $H$ is related to the quantum Hamiltonian $\hat{H}$ by

$$H(x)=<x|\hat{H}|x>. \tag{44}$$

Then the Hamiltonian of the system, the environment, and the interaction between them are expressed as

$$H_s(\bar{\eta},\eta)=<\eta_l|\hat{H}_s|\eta_l>=\sum_{l=1}^{2}<\eta_l|\hbar\omega_l J_{l+}J_{l-}|\eta_l>, \tag{45}$$

$$H_E(\bar{z},z)=<z_k|\hat{H}_E|z_k>=\sum_k<z_k|\hbar\omega_k b_k^{\dagger}b_k|z_k>, \tag{46}$$

$$H_I(\bar{\eta},\eta,\bar{z},z) =< \eta_l, z_k|\hat{H}_I|\eta_l, z_k >= \sum_{l,k} < \eta_l, z_k|\hbar(g_{lk}J_{l+}b_k + g_{lk}^* J_{l-}b_k^\dagger)|\eta_l, z_k >. \quad (47)$$

In the basis $|j,m>$ the spin coherent state take the form

$$|\eta_l >= (1+|\eta|^2)^{-j} \sum_{m=-j}^{j} \binom{2j}{j+m}^{\frac{1}{2}} \eta^{j+m}|j,m>, \quad (48)$$

thus

$$J_+|\eta_l >= (1+|\eta|^2)^{-j} \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j+m+1)!(j-m-1)!}\right]^{\frac{1}{2}} (j+m+1)\eta^{j+m}|j,m+1>, \quad (49)$$

and

$$J_-|\eta_l >= \eta(1+|\eta|^2)^{-j} \sum_{m=-j+1}^{j} \left[\frac{(2j)!}{(j+m-1)!(j-m+1)!}\right]^{\frac{1}{2}} (j-m+1)\eta^{j+m-1}|j,m-1>, \quad (50)$$

And

$$< \eta|J_+|\eta >= (1+|\eta|^2)^{-j} + (1+|\eta|^2)^{j-1}2j\eta\bar{\eta},$$

$$< \eta|J_-|\eta >= (2j+1)(1+|\eta|^2)^{-j} - (1+|\eta|^2)^{j-1}2j\eta\bar{\eta}, \quad (51)$$

$$< \eta|J_+J_-|\eta >= 2j\eta\bar{\eta}(2j+2j|\eta|^2 - (2j-1)(\eta\bar{\eta})^2)(1+|\eta|^2)^{-2},$$

Thus

$$< z|b^\dagger = (b|z>)^\dagger =< z|\bar{z},$$

Then

$$H_s(\bar{\eta},\eta) = \hbar\sum_{l=1}^{2} \omega_l (1+|\eta|^2)^{-2}[4j^2\eta\bar{\eta}(1+|\eta|^2 - \eta\bar{\eta}) + \eta\bar{\eta}]$$

$$H_s(\bar{z},z) = \sum_k \omega_k \bar{z}_k z_k \quad (52)$$

$$H_I(\bar{\eta},\eta,\bar{z},z) = \hbar\sum_{l,k} (1+|\eta_l|^2)^{j-1}[g_{lk}z_k((2j+1)(1+|\eta_l|^2) - 2j\eta_l\bar{\eta}_l) + g_{lk}^*\bar{z}_k(2j\eta_l\bar{\eta}_l + (1+|\eta_l|^2)^{-(2j+1)})],$$

where $\bar{z}$ and $\bar{\eta}$ denote the complex conjugate of $z$ and $\eta$, respectively. We have

$$\partial_z H_I(\bar{\eta},\eta,\bar{z},z) = \hbar g\,(1+|\eta|^2)^{-1}(2j+|\eta|^2+1),$$

$$\partial_{\bar{z}} H_I(\bar{\eta},\eta,\bar{z},z) = \hbar g\,(1+|\eta|^2)^{-1}(2j\eta\bar{\eta}+|\eta|^2+1).$$

And

$$\partial_z H_E(\bar{z},z) = \hbar\omega\bar{z},$$

$$\partial_{\bar{z}} H_E(\bar{z}, z) = \hbar\omega z.$$

The equations of movement are

$$\dot{z} = -\frac{i}{g(\eta,\bar{\eta})}[\omega Z + g\,(1+|\eta|^2)^{-1}(2j+|\eta|^2+1)], \quad (53)$$

and

$$\dot{\bar{z}} = \frac{i}{g(\eta,\bar{\eta})}[\omega \bar{Z} + g\,(1+|\eta|^2)^{-1}(2j\eta\bar{\eta}+|\eta|^2+1)]. \quad (54)$$

The solutions are

$$Z(\tau) = Z_i e^{-\frac{i\omega}{g(\eta,\bar{\eta})}\tau} - ig\int_0^\tau d\tau'\, e^{-\frac{i\omega}{g(\eta,\bar{\eta})}(\tau-\tau')}(1+|\eta|^2)^{-1}(2j+|\eta|^2+1), \quad (55)$$

and

$$\bar{Z}(\tau) = \bar{Z}_f e^{-\frac{i\omega}{g(\eta,\bar{\eta})}(t-\tau)} + ig\int_\tau^t d\tau'\, e^{-\frac{i\omega}{g(\eta,\bar{\eta})}(\tau-\tau')}(1+|\eta|^2)^{-1}(2j\eta\bar{\eta}+|\eta|^2+1), \quad (56)$$

and also

$$Z(\tau) = -\frac{i\omega}{g(\eta,\bar{\eta})} Z_i e^{-\frac{i\omega}{g(\eta,\bar{\eta})}\tau} - g\frac{\omega}{g(\eta,\bar{\eta})}\int_0^\tau d\tau'\, e^{-\frac{i\omega}{g(\eta,\bar{\eta})}(\tau-\tau')}(1+|\eta|^2)^{-1}(2j+|\eta|^2+1). \quad (57)$$

We have $S_s$, $S_E$ and $S_I$ are the actions corresponding to the system, the environment and the interaction Hamiltonian $H_s$, $H_E$ and $H_I$ respectively. Then

$$S_E[\bar{z}, z] = \sum_k\{-i\hbar\bar{z}_k z_k(t) + \int_0^t d\tau[i\hbar\bar{z}_k\dot{z}_k(\tau) - H_E(\bar{z}, z)]\}. \quad (58)$$

In the same way we can found

$$S_s[\bar{\eta}, \eta] = \int_0^t \left[i\hbar j\frac{\bar{\eta}\dot{\eta} - \dot{\bar{\eta}}\eta}{1+\bar{\eta}\eta} - H_s(\bar{\eta}, \eta)\right] d\tau + B, \quad (59)$$

where

$$B = -\, i\hbar j \ln[(1+\bar{\eta}_f\eta(t))(1+\bar{\eta}(0)\eta_i)], \quad (60)$$

and

$$S_I[\bar{\eta}, \eta, \bar{z}, z] = -\int_0^t d\tau\, H_I(\bar{\eta}, \eta, \bar{z}, z), \quad (61)$$

can write as

$$S_I[\bar{\eta}, \eta, \bar{z}, z] = -\int_0^t d\tau\, \hbar \sum_{l,k}(1+|\eta_l|^2)^{j-1}\,[g_{lk} z_k((2j+1)(1+|\eta_l|^2) - 2j\eta_l\bar{\eta}_l)$$
$$+ g_{lk}^* \bar{z}_k\big(2j\eta_l\bar{\eta}_l + (1+|\eta_l|^2)^{-(2j+1)}\big]. \quad (62)$$

We have

$$K = \int D_{path} \exp(\frac{i}{\hbar} Action)$$

$$= \int D_{path} \exp(\frac{i}{\hbar} S(path)),$$

with

$$DZD\bar{Z} = \prod_k e^{-\bar{z}_k z_k} d\,\bar{z}_k dz_k,$$

$$D\eta D\bar{\eta} = \prod_l d\eta_l d\bar{\eta}_l\,(1+|\eta_l|^2)^{-2}.$$

The propagator is

$$K(\bar{\eta}_f, \eta'_f; t|\eta_i, \bar{\eta}'_i; 0) = \int d\mu(z_f)\, d\mu(z_i) d\mu(z'_i) < \eta_f, z_f; t|\eta_i, z_i; 0 >$$

$$\times \rho_E(\bar{z}_i, z'_i; 0) \quad < \eta'_i, z'_i; 0|\eta'_f, z_f; t >, \qquad (63)$$

where

$$< \eta_f, z_f; t|\eta_i, z_i; 0 >= \int DzD\bar{z}D\eta D\bar{\eta} \exp\{i(\,S_s[\bar{\eta}, \eta] + S_I[\bar{\eta}, \eta, \bar{z}, z] + S_E[\bar{z}, z])\}. \qquad (64)$$

Then, the final form of propagator is

$$K(\bar{\eta}_f, \eta'_f; t|\eta_i, \bar{\eta}'_i; 0) = \int D^2\eta D^2\eta' \exp\{\,\mathrm{j}\ln\left[\left(1+\bar{\eta}_f\eta(\tau)\right)(1+\bar{\eta}(0)\eta_i)\right]]$$

$$-\mathrm{j}\ln\left[\left(1+\eta'_i\bar{\eta}'(\tau)\right)\left(1+\eta'(0)\bar{\eta}'_f\right)\right] - \int_0^{\mathrm{t}} \mathrm{j}\frac{\bar{\eta}\dot{\eta}-\dot{\bar{\eta}}\eta}{1+\bar{\eta}\eta} - \mathrm{j}\frac{\bar{\eta}'\dot{\eta}'-\dot{\bar{\eta}}'\eta'}{1+\bar{\eta}'\eta'}$$

$$+\hbar\omega(1+|\eta|^2)^{-2}[4j^2\eta\bar{\eta}(1+|\eta|^2-\eta\bar{\eta})+\eta\bar{\eta}]$$

$$+\hbar\omega(1+|\eta'|^2)^{-2}[4j^2\eta'\bar{\eta}'(1+|\eta'|^2-\eta'\bar{\eta}')+\eta'\bar{\eta}']\,\}$$

$$\times \exp\{\int_0^t d\tau \,\int_0^\tau d\tau'\mu(\tau-\tau') \times$$

$$[(1+|\eta|^2)^{-1}(2j+|\eta|^2+1) \times (1+|\eta'|^2)^{-1}(2j\eta'\bar{\eta}'+|\eta'|^2+1)]$$

$$+\mu^*(\tau-\tau')[\{(1+|\eta'|^2)^{-2}(2j+|\eta'|^2+1)(2j\eta'\bar{\eta}'+|\eta'|^2+1)\}$$

$$-\{(1+|\eta|^2)^{-2}(2j+|\eta|^2+1)(2j\eta\bar{\eta}+|\eta|^2+1)\}]\}. \qquad (65)$$

In this work we have presented a concrete application of the semi-classical spin propagator. In general the calculation involves a classical trajectory $(\eta(t), \bar{\eta}(t))$, that starts at $\eta = \eta_i$ and ends at $\bar{\eta} = \eta^*_f$ after a time $t$. Under these too stringent conditions, the only way to find a classical trajectory is by allowing the variable $\bar{\eta}(t)$ to be different from the complex conjugate of $\eta(t)$ (which can be denoted by $\eta^*(t)$. We must therefore find a trajectory in $\mathbb{C}^2$ that satisfies the boundary conditions $\eta(0)$=$\eta_i$ and $\bar{\eta}(t) = \eta^*_f$ .

## 3. CONCLUSION

The coherent state spin path integral is an ingredient in the implementation of some quantum information protocols. The semi classical approximation to the spin propagator:

$$K(\eta_i, \bar{\eta}_f; t) = < \eta_f | e^{-iHT/\hbar} | \eta_i >,$$

may sometimes lead to an expression to the geometric phase in terms only of classical quantities.

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*** Corresponding author: SEDDIK Sanaa**

LPHE-Modeling and Simulation, Faculty of Sciences, University Mohammed V, Rabat, Morocco

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