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# A Theoretical study of hydrodynamic Instability and Quantum turbulence in atomic Bose Einstein Condensate BECs) and evaluation of incompressible and compressible kinetic energy in 2D non-uniform BECs and spin dependent interaction energy for spinor BECs

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**Abstract:** Using the theoretical formalism of **M Tsuboto [arXiv: 1312.0092v2 (cond-mat.quant-gas), 7 Jan 2014], K Fujimoto and M. Tsubota {Phys Rev A 88, 063628 (2013 )}, T-L Hong etal. {Phys. Rev. A80, 023618 (2009)] and M Tsubota {J. Phys. Soc. Jpn. 77, 111006 (2008)},** we have theoretically studied hydrodynamic instability and turbulence in quantum fluids. We have reviewed the physics of classical turbulence (CT) in super fluid helium and quantum turbulence (QT) in atomic BECs, 2D non-uniform BECs, Oscillating BECs and spinor BECs respectively. It has been observed that in the case of classical turbulence (CT) as seen in super fluid helium can be well studied with the help of vortex filament model and use of Biot-Savart law. The quantum turbulence (QT) in the above quantum fluids can be well studied with the help of two coupled GP equation and scaling law analysis. Our evaluated results of incompressible kinetic energy $E^{i}{}_{kin}$ and compressible kinetic energy $E^{c}{}_{kin}$ in the case of 2D non uniform BECs increase and decrease as a function of time t(s), We have also observed that for each time $E^{c}_{kin} > E^{i}_{kin}$. Our theoretically obtained results of incompressible kinetic energy spectrum for two different range show different power-law behaviour. For 2<k<4, it follows $k^{\frac{-5}{3}}$ power law which is consistent with Kolmogorov result. For k>10, it obeys $k^{-4}$ power law as suggested by Soffman. Between these two range of spectral scaling, there is a transitional zone 4<k10, $E^{i}_{kin}$ (k) exhibits no scaling behaviour, Our evaluated results of enstrophy z(k) also increase and decrease with time and follows $k^{\frac{-1}{3}}$ power law in the first range and $k^{-2}$ for the second range respectively. In the case of spinor BECs, our evaluated results of spin dependent interaction energy( as a function of $ka_h$) also show -7/3 power law behaviour. Similar behaviour has been noticed for the time development of exponent $n_p$. Such type of scaling behaviour is observed in the case of magnetic spin glass system. But there is a difference between spin-glass behaviour of spinor BECs and magnetic spin glass. In case of spinor BECs, there is no long range order and spins are randomly oriented. This has been noticed in our evaluation of time development of the spin glass order parameter q(t) which grows as the spin turbulence develops a -7/3 power law. Our theoretically evaluated results are in good agreement with other theoretical workers.

The paper reviews the recent important development in the field of CT and QT observed in varieties of BECs. The evaluation has been performed on the basis of quantum hydrodynamics and power law scaling analysis. The results obtained for 2D non uniform BECs and spinor BECs will be quite useful in order to understand the fundamental physcs of turbulence.

**Keywords:** Classical turbulence (CT), Quantum turbulence (QT), Hydrodynamics instability, Vortex filament model, Two coupled GP equation, Biot-Savart law, Atomic BECs, 2D non-uniform BECS, Oscillaing BECs, Spinofr BECs, Magnetic spin glass system, Order parameter for the spin-glass, Incompressible and compressible kinetic energy, Enstrophy, Quantized vortex, Vorticity, Topological defects, Kolmogorov spectrum.

## INTRODUCTION

Bose Einstein condensation is often considered to be a macroscopic quantum phenomenon. This is because bosons constituting a system occupy the same single-particle ground state below a critical temperature. Bose Einstein condensation forms a macroscopic wave function (order parameter) which extends over the entire system. As a direct result, quantized vortices appear in the Bose condensed system. A quantized vortex is a vortex of in viscid super flow, and any rotational motion of a super fluid is sustained by quantized vortices. Hydrodynamics dominated by (QHD). The turbulence comprised by quantized vortices is known as quantum turbulence (QT). The studies of quantized vortices originally started in 1950s using super fluid $^4$He. After that much theoretical, numerical and experimental efforts has been devoted in this field.

Research work in the field of QHD and QT has been tended toward in new direction[1,2] . The new direction is in the field of low temperature physics, studying super fluid helium. The field of study started in order to understand the relationship between quantum turbulence (QT) and classical turbulence (CT). The energy spectrum of fully developed CT is known to obey the Kohmogorov law in the inertial range[3-5]. Recent experimental and numerical studies support Kohmogorov spectrum in QT. Another new direction is the realization of Bose Einstein Condensation in trapped atomic gases[6,7]. In this field also the existence of super fluidity, quantized vortices in trapped atomic gases have been created and observed in BECs. Atomic BEC has several advantages over super fluid helium. The most important is that the modern optical techniques enable the direct control and the visualization of condensates.

A quantized vortex is a topological defect characteristic of a Bose Einstein condensate and is different from a vortex in a classical viscous fluid. First, its circulation is quantized and conserved, in contrast to a classical vortex whose circulation can have any value and not conserved. Second, a quantized vortex is a vortex of inviscid super flow. Thus, it can not decay by the viscous diffusion of vorticity that occurs in a classical fluid. Third, the core of a quantized vortex is very thin on the order of coherence length, which is in the order of few angstroms in super fluid $^4$He and submicron in atomic BECs. Since the vortex core is thin and does not decay by diffusion, it is possible to identify the position of a quantized vortex in the fluid. These properties make a quantized vortex more stable and definite than a classical vortex.

In this paper, using the theoretical formalism of M Tsubota[8], K Fujimoto and M Tsubota[9], T-L Hong etal[10] and M. Tsubota[11], we have theoretically studied hydrodynamic instability and turbulence in quantum fluids. First, we have reviewed classical turbulence (CT) and then quantum turbulence (QT) in these studies and used quantum hydrodynamics. In quantum fluids, we have studied the QT in 2D non-uniform BEC, oscillating BEC, atomic BEC and spinor BEC respectively. We have observed that CT in the case of super fluid $^4$He can be well studied with the help of Vortex model[12] and the use of Biot-Savart law[13]. In the case of QT for 2D non-uniform BECs, we have evaluated time development of incompressible kinetic energy( $E^{i}{}_{kin}$ ) and compressible energy ( $E^{c}{}_{kin}$ for fully turbulent 2D non-uniform BECs. Our theoretically evaluated results show that $E^{c}{}_{kin}$ is greater than $E^{i}{}_{kin}$ and both energies decrease and increase with time. Our evaluated result of a physical parameter called enstrophy also increase and decrease with time. Our evaluated results of incompressible kinetic energy spectrum of the flow at t=100s takes a power law for the two ranges of spectral scaling. We have also shown the experimental results[14] of the emergence of QT in an oscillating BEC. They have calculated the aspect ratio of the cloud during time of flight (TOF) measurement for normal BEC and oscillating QT BEC. They observed that both the aspect ratios increase with time.

Our evaluated results of spin dependent interaction energy as a function of spinor radius follow a -7/3 power law. Our evaluated result of time dependent of the exponent $n_p$ of the spectrum of spin dependent interaction energy show a spin-glass like order which grows as the spin turbulence develops a -7/3 power law. This result gives a signature of spin quantum turbulence in spinor BEC[15]. Our evaluated results of time dependence of the order parameter q(t) increases as the exponent $n_p$ (t) approaches -7/3 for spin glass behaviour. These results also confirm that the system of spinor BEC differs from a magnetic system which also yields spin-glass behaviour[16]. Our evaluated results are in good agreement with other theoretical workers[17-21].

## MATHEMATICAL FORMULA USED IN THE STUDY

First of all, we shall discuss the model of quantum turbulence. There are generally two models available for the numerical simulation of quantum turbulence. One is the vortex filament model[12] and the other is Gross-Pitaevskii (GP) model[22].

**Vortex Filament (VF) model:** As one knows that a quantized vortex has quantized circulation. The vortex core is extremely thin, usually much smaller than other characteristic length scales of the vortex motion. These properties allow a quantized vortex to be represented as a vortex filament. In classical fluid dynamics, the vortex filament (VF) model is only an idealization. However, the VF model is accurate and realistic for a quantized vortex in a super fluid helium. The VF model represents a quantized vortex as a filament passing through the fluid, having a definite direction corresponding to its vorticity[23]. The velocity at point **r** due to filament is given by

$$V^{\rightarrow}{}_{s}(r^{\rightarrow}) = \frac{\kappa}{4\pi}\int_{c}\frac{(s^{\rightarrow}{}_{1} - r^{\rightarrow})}{\left|s^{\rightarrow}{}_{1} - r^{\rightarrow}\right|^{2}}ds^{\rightarrow}{}_{1} \qquad \text{… (1)}$$

The vortex moves with the super fluid velocity. At finite temperature, mutual friction occurs between the vortex core and the normal flow. A numerical simulation method on this model has been given in more detail in the paper [24-27]. Later on it was also shown that this is accompanied by the emission of sound waves having wavelengths on the order of healing length[28,29].

**The Gross-Pitaevskii (GP) model:** In a weakly interacting Bose system, the macroscopic wavelength $\psi(x^{\rightarrow}, t)$ appears as an order parameter of the Bose Einstein condensation obeying the GP equation[22]

$$i\hbar\frac{\partial\psi(x^{\rightarrow}, t)}{\partial t} = [-\frac{\hbar^{2}}{2m}\nabla^{2} + g\left|\psi(x^{\rightarrow}, t) - \mu\right]\left|\psi(x^{\rightarrow}, t)\right. \qquad \text{… (2)}$$

Here, g refers to the interaction parameter and $\mu$ is the chemical potential. Writing $\psi = \left|\psi\exp(i\theta)\right|$, the square amplitude $\left\|\psi\right\|^{2}$ is the condensate density and the gradient of the phase $\theta$ gives the super fluid velocity $v^{\rightarrow}{}_{s} = (\hbar/m)\nabla\theta$ corresponding to frictionless flow of the condensate. The only characteristic length of GP model is coherence length defined by $\xi = \hbar/((2mg)^{\frac{1}{2}}\left|\psi)\right|$, which determines the vortex core size.

Now the question arises among VF and CP models which model should be used as for realistic simulation. The criterion comes from advantages and disadvantages of each model. The VF model neglects the core structure of quantized vortices. Hence, it can describe well the dynamics of quantized vortices as for as the coherence length is much smaller than the other characteristic lengths are concerned. This includes radius of curvature of vortex lines, the mean distance l between the vortices, and the system size. However, it cannot explain phenomena related to vortex cores, such as reconnection, nucleation and annihilation. Thus, the VF model is usually used for simulation in super fluid $^{4}He$ and super fluid $^{3}He$-B. On the other hand, the GP model can describe not only the dynamics of vortices but also the phenomenon related to vortex cores. This is quite useful for trapped atomic BECs in which the coherence length is usually much smaller than the system size. The GP model is not quantitatively applicable to super fluid helium which is strongly correlated system. However, the GP model has long been studied as the simplest model of BEC and super fluidity in BEC.

**Recent studies of quantum turbulence (QT):** Recent studies of QT has been advanced over older studies which were chiefly limited to thermal counter flow in $^4$He, which has no analogue with classical traditional turbulence. The new studies on QT are focussed on a comparison between QT and CT. The second reason is the realization of BECs for which modern optical techniques enable the direct control and visualization of the condensate and even can change the interaction. Such direct control is impossible in other quantum condensate like super fluid helium and superconductors.

**Energy spectrum of quantum turbulence:** The energy spectrum of QT obeys Kolmogorov law[30,31]

$$E(k) = C\varepsilon^{\frac{2}{3}} k^{-(\frac{5}{3})} \qquad \dots (3)$$

Here, $\varepsilon$ is the energy flux and k is the wave number. The Kolmogorov law is the most important statistical law in turbulence. Since Qt at finite temperatures is much affected by normal fluid and complicated, therefore one studies QT at zero temperature.

In QT vortices form complicated tangled structure. If the tangled is assumed to be homogeneous and isotropic there are two characteristic length scales. One is the mean inter vortex spacing $l = L^{\frac{-1}{2}}$ with vortex line length density L, and the other is the coherence length $\xi$ corresponding to the size of vortex core. Generally, $l \,\square\, \xi$. One defines characteristic wave number $k_l = 2\pi / l$ and $k_\xi = 2\pi / \xi$. The energy spectrum depends on the scale and its properties change at about the scale of the mean inter-vortex spacing l. when $k \,\square\, k_l = 2\pi / l$, a Richardson cascade of quantized vortices transfers energy from large to small scales maintaining the Kolmogorov spectrum $E(k) = C\varepsilon^{\frac{2}{3}} k^{\frac{-5}{3}}$. When k $\square\, k_l$, energy is transferred by the Kelvin-wave cascade, which is a nonlinear interaction between Kelvin waves of different wave numbers. Eventually, energy is dissipated at scales of $\xi$ by the radiation of phonons.

**QT created by vibrating structures:** Recently, vibrating structures such as discs, spheres, grids and wires have been widely used for research into QT[32,33]. Despite detailed differences between the structures considered, the experiments show some surprisingly common phenomena. In order to characterize the transition to turbulence, Fujiama etal[34] studied the drag force. The drag force acting on an object in a uniform flow is generally represented by

$$F_D = (1/2) C_D \rho A U^2 \qquad \dots (4)$$

Here, $C_D$ is the drag coefficient, $\rho$ is the fluid density, A is the projection area of the object normal to the flow, and U is the flow velocity. At low Reynolds number Stokes drag force acts on the object, which is proportional to the magnitude of U, with the result that $C_D$ is inversely proportional to U. When the flow becomes turbulent at high Reynolds number, $C_D$ is order of unity. One can study the analogy between QT and CT with this.

**Developments in atomic BECs:** The achievements of Bose Einstein condensation in trapped atomic gases has stimulated intense experimental and theoretical activity in modern physics[35,36]. This innovation has provided another important stage of quantum hydrodynamics because of the peculiar feature of the system. First, this system is a dilute Bose gas, thus the GP equation (2) gives a quantitatively accurate description of the static and dynamic properties of the atomic BEC. The vortex structure and dynamics can be discussed by a more fundamental approach than with super fluid helium. Secondly, the finite size effect due to the trapping potential and the associated density in homogeneity yield new characteristics of vortex physics. Thirdly, the manipulation of condensate wave function via external fields provides a versatile scheme to control the BEC. Fourthly, the vortex cores can be directly visualized through the observed density profile by a time-of-flight (TOF) technique.

Observing quantized vortices has been certainly an important smoking gun of super fluidity in atomic BECs. Many experimental and theoretical works have been devoted to the issue of quantized vortices in this system[37-40]. Generally, two kinds of cooperative phenomena of quantized vortices are seen. (a) a vortex array under rotation (b) vortex tangle in QT. Both phenomena have been intensively studied in the field of super fluid helium, while most studies of quantized vortex in atomic BECs have been limited for vortex array under rotation.

**Two-Dimensional Quantum Turbulence in A Non-Uniform BECS:**

**Numerical integration of the Gross-Pitaevskii equation:** Now, one can study the turbulent dynamics of the 2D BEC confined in a harmonic trap by numerical integration of time dependent Gross-Pitaevskii equation (2) written in the following dimensionless form

$$i\frac{\partial\psi(\rho^{\rightarrow},t)}{\partial t}=[-\frac{\nabla^2}{2}+\frac{(x^2+\lambda^2{}_y y^2)}{2}+g\left|\psi(\rho^{\rightarrow},t)\right|^2]\psi(\rho^{\rightarrow},t) \qquad \ldots (5)$$

Here, $\rho=(\mathrm{x},\mathrm{y})$ and the condensate wave function $\psi(\rho^{\rightarrow},t)$ is normalized by

$$\int\left|\psi(\rho^{\rightarrow},t)\right|d\rho=1 \qquad \ldots (6)$$

The coupling strength is expressed by $g=4\pi\bar{a}N$, where N is the total number of atoms in the condensate and $\bar{a}$ is the s-wave scattering length in 2D space. The harmonic trap is given by $\mathrm{V}(\rho^{\rightarrow})=(1/2)\mathrm{m}\,\omega^2(x^2+\lambda^2{}_y y^2)$ .To numerically integrate equation (5), one uses the method of lines with spatial discretization by accurate Fourier pseudo-spectral method and time integration by adaptive Runge-Kutta method of orders 2 and 3 (RK23). Initially, one placed isotropic condensate ($\lambda_y=1$) with a Thomas –Fermi radius (TF) $\mathrm{R}_{TF}=\sqrt{2\mu}$ =6, where $\mu$ is the chemical potential of the condensate.

Now, one applies the spectral scaling approach to identify the occurrence of turbulence in the trapped BEC. For this, one expresses the condensate wave function in the form of Madelung transformation namely $\psi(\rho^{\rightarrow},t)=\sqrt{n(\rho^{\rightarrow},t)}e^{i\varphi(\rho^{\rightarrow},t)}$ .Substituting the above into equation (5), one obtain the continuity equation and Euler equation respectively

$$\frac{\partial}{\partial t}n+\nabla.(nu^{\rightarrow})=0 \qquad \ldots (7)$$

$$\frac{\partial}{\partial t}u^{\rightarrow}+u^{\rightarrow}.\nabla u^{\rightarrow}=-\nabla Q \qquad \ldots (8)$$

Here, $u^{\rightarrow}(\rho^{\rightarrow},t)=\nabla\varphi(\rho^{\rightarrow},t)$ is the velocity field of the super-flow and $\mathrm{Q}=\mathrm{gn}+\mathrm{V}-(\nabla^2\sqrt{n})/2\sqrt{n}$. The total energy functional is calculated in terms of n and **u,** that can be expressed as the sum of four terms

$$E_{tot} = \int [(1/2)\left|\nabla\sqrt{n}e^{-i\varphi}\right|^2 + \frac{(x^2+\lambda^2{}_y y^2)}{2} n + (1/2)gn^2]\,\mathrm{d}\,\rho^{\rightarrow} \qquad \ldots (9)$$

$$= E_{kin} + E_q + E_{tr} + E_{\text{int}} \qquad \ldots 9(b)$$

Here, $E_{kin}$ is the super fluid kinetic energy, $E_q$ is the quantum pressure energy, $E_{tr}$ is the trap energy and $E_{int}$ is the interaction energy respectively. The algebraic expression for these energies are given by

$$E_{kin} = (1/2)\int \left|\sqrt{n}u^{\rightarrow}\right|^2 d\rho^{\rightarrow} \qquad \ldots (10a)$$

$$E_q = (1/2)\int (\nabla\sqrt{n})^2 d\rho^{\rightarrow} \qquad \ldots (10b)$$

$$E_{tr} = (1/2)\int n(x^2+\lambda^2{}_y y^2)\,\mathrm{d}\,\rho^{\rightarrow} \qquad \ldots (10c)$$

$$E_{\text{int}} = (1/2)\int gn^2 d\rho^{\rightarrow} \qquad \ldots (10d)$$

One decomposes the vector field $\sqrt{n}u^{\rightarrow}$ into incompressible and compressible parts i.e., $\sqrt{n}u^{\rightarrow} = (\sqrt{n}u^{\rightarrow})^i + (\sqrt{n}u^{\rightarrow})^c$ , where $\nabla.(\sqrt{n}u^{\rightarrow})^i = 0$ and $\nabla x(\sqrt{n}u^{\rightarrow})^c = 0$. Let $(\sqrt{n}u^{\rightarrow})^i = \nabla x A^{\rightarrow}$ and $(\sqrt{n}u^{\rightarrow})^c = \nabla\Theta$, where **A** and $\Theta$ are vector and scalar potential of the field $\sqrt{n}u^{\rightarrow}$ such that $\sqrt{n}u^{\rightarrow} = \nabla x A^{\rightarrow} + \nabla\Theta$. Now, one gets Poisson equation for the scalar field

$$\nabla^2\Theta = \nabla.\sqrt{n}u^{\rightarrow} \qquad \ldots (11)$$

This equation is numerically solved by Fourier pseudo-spectral method. Once the scalar potential $\Theta$ is numerically solved, the vector potential **A** can be determined hence the field components $(\sqrt{n}u^{\rightarrow})^i$ and $(\sqrt{n}u^{\rightarrow})^c$. Thus, the incompressible and compressible kinetic energies are defined by $E^{i,c}{}_{kin} = (1/2)\int d\rho^{\rightarrow}[(\sqrt{n}u^{\rightarrow})^{i,c}]^2$. This follows that $E_{kin} = E^i{}_{kin} + E^c{}_{kin}$. Physically, $E^i{}_{kin}$ and $E^c{}_{kin}$ corresponds to kinetic energies of swirls and sound waves in the super-flow respectively. Now, one calculates of the enstrophy of super flow for the comparison with the results of 2D classical turbulence (CT). Classically, the enstrophy Z is defined as $Z = (1/2)\int \left|\omega^{\rightarrow}\right|^2 d\rho^{\rightarrow}$, where $\omega^{\rightarrow}$ is the vorticity vector. We have theoretically evaluated results of time evaluation of $E^i{}_{kin}$ and $E^c{}_{kin}$ in table T4 and time evaluation of enstrophy in table T5 respectively.

**Emergence of quantum turbulence in an oscillating Bose-Einstein condensate (BECs):** There is an experimental observation[14] of vortex tangles in an atomic BEC of $^{87}$Rb atoms when an external oscillatory perturbation is introduced in the trap. The vortex tangle configuration is a signature of presence of turbulent regime of the cloud. One also shows that this turbulence cloud suppresses the aspect ratio inversion typically observed in quantum degenerate bosonic gases during free expansion. Instead, the cloud expands keeping the ratio between their axis constant. Turbulent in an atomic super fluids may constitute an alternative system to investigates decay mechanism as well as to test fundamental theoretical aspects in this flow, We have shown the experimental results of the emergence of quantum turbulence in an atomic quantum gases by taking the aspect ratio of the normal BEC and oscillatory BEC. The results are shown in table T6.

**Quantum turbulence in multi-component BECs:** Multi-component BECs yield a rich variety of super fluid dynamics[41,42]. The macroscopic wave function $\psi_j\left(x^{\rightarrow},t\right)(j=1,2)$ of two component BECs at zero temperature obey the coupled GP equation[22]

$$i\hbar\frac{\partial\psi_j\left(x^{\rightarrow},t\right)}{\partial t}=[-\frac{\hbar^2}{2m_j}\nabla^2+\Sigma_k g_{jk}\left|\psi_k(x^{\rightarrow},t)\right|^2-\mu_j]\psi_j\left(x^{\rightarrow},t\right) \qquad \ldots (12)$$

Here $m_j$ and $\mu_j$ are the particle mass and chemical potential of jth component. When the intra-component interaction $g_{jj}$ and inter-component interaction $g_{jk}$ satisfy the relation $g_{11}g_{22}>g^2{}_{12}$, the two BECs like to be miscible.

When two component BECs coexist with a relative velocity, they exhibit dynamic instability above a critical relative velocity[43]. This phenomenon is known as a counter-super flow instability (CST). The CSI can be understood from the Bogoliubov spectrum for a uniform two-component BEC with a relative velocity[44]. Takeuchi etal[45,46] suggested that the nonlinear dynamics triggered by CSI generates a binary QT.

The important point is to know how two BECs interact. Each BEC exhibits super fluidity respectively. An optical trap removes the restriction of the confinable hyperfine spin states of atoms, this realizing new multi-component BECs with internal degree of freedom referred to as spinor BEC[47,48]. In contrast to the two-component BECs, inter atomic interactions allow for a coherent transfer of population between different hyperfine spin states (spin-exchange collision), which yields a fascinating physics different from two-component BECs.

One considers a spin-1 spinor BEC at zero temperature. The macroscopic wave function $\psi_m$ with the magnetic quantum number $m(m=1.0,-1)$ obey the GP equation[49,50]

$$i\hbar\frac{\partial}{\partial t}\psi_m=[(-\frac{\hbar^2}{2M}\nabla^2+V)\psi_m+\sum_{n=-1}^{1}\{-g\mu_B(B^{\rightarrow}*S^{\rightarrow})_{nm}+q(B^{\rightarrow}*S^{\rightarrow})^2{}_{nm}\}\psi_n$$

$$+c_0 n\psi_m+c_1\sum_{n=-1}^{1}s*S_{nm}{}^{\rightarrow}\psi_n \qquad \ldots (13)$$

Here, V and **B** are trapping potential and magnetic field. The parameters M, g, $\mu_B$ and q are the mass of a particle, the Lande g factor, the Bohr magneton and a coefficient of the quadratic Zeeman effect respectively. The total density n and the spin density vector $s_i(x,y,z)$ are given by $n=\sum_{m=-1}^{1}\left|\psi_m\right|^2$ and $s_i=\sum_{m,n=-1}^{1}\psi^*{}_m(S^{\rightarrow}{}_i)_{mn}\psi_n$ with the spin-1 matrices $(S^{\rightarrow}{}_i)_{mn}$. The parameters $c_0$ and $c_1$ are the coefficients of the spin-independent and spin dependent interactions. One focuses the spin-dependent interaction energy $E_{spin}=\frac{c_1}{2}\int s^{\rightarrow 2}dr^{\rightarrow}$, whose coefficient $c_1$ determines whether the system is ferromagnetic ($c_1$<0) or anti-ferromagnetic ($c_1$>0).

## DISCUSSION OF THE RESULTS

In this paper, using the theoretical formalism of M. Tsubota[8], K Fujimoto and M tsubota[9], T-L Hong etal.[10] and M.Tsubota[11], we have theoretically studied hydrodynamic instability and turbulence in quantum fluids. First, we have reviewed the work done in classical turbulence (CT) and quantum turbulence (QT) with the use of quantum hydrodynamic relations in these studies. In quantum fluids, we have outlined the classical turbulence studied with the help of vortex model[12] using Biot-Savart law[13] in superfluid helium, QT in atomic BEC, 2D non-uniform BEC, QT in oscillating BEC and finally QT in spinor BEC. In **table T1**, we have shown the numerical results of simulation of the vortex filament model for computing a box of 0.1x0.1x0.1cm$^3$. This result was obtained by Adachi etl.[51] based on ghe calculation by vortex model[12] with the use of Biot-Savart law for classical turbulence in super fluid $^4$He. This law account for the inter vortex interaction to obtain statistical steady states,[52]. Here the vortex line density L(cm$^{-2}$) is calculated as a function of time for different counter-flow velocities $v_{ns}$. Vortex line density (VLD) is the total length of the vortex lines per unit volume grows towards a statistically steady temperature. They observed that the vortex line density decrease with the decrease of counter flow velocity.

**Table T1:** This table shows the results of vortex line density ( $L(cm^{-2})$ as a function of time t(s) for four different counter flow velocities $v_{ns}(cm/s)$, results are the numerical simulation of the vortex filament model by the full Biot-Savart law for classical turbulence for super fluid $^4$He, results were calculated by Adachi *et al.*[51]

| t(s) | ← $L(cm^{-2})$ → | | | |
|---|---|---|---|---|
| | $v_{ns}$ =0.572 cm/s | $v_{ns}$ =0.476 cm/s | $v_{ns}$ =0.354 cm/s | $v_{ns}$ =0.286cm/s |
| 0.0 | 0.00 | 0.00 | 0.000 | 0.000 |
| 1.0 | 0.24x10$^4$ | 0.16x10$^4$ | 0.05x10$^4$ | 0.002x10$^4$ |
| 1.5 | 0.32x10$^4$ | 0.22x10$^4$ | 0.12x10$^4$ | 0.012x10$^4$ |
| 2.0 | 0.38x10$^4$ | 0.27x10$^4$ | 0.15x10$^4$ | 0.10x10$^4$ |
| 2.5 | 0.43x10$^4$ | 0.33x10$^4$ | 0.19x10$^4$ | 0.13x10$^4$ |
| 3.0 | 0.45x10$^4$ | 0.37x10$^4$ | 0.23x10$^4$ | 0.15x10$^4$ |
| 3.5 | 0.48x10$^4$ | 0.39x10$^4$ | 0.27x10$^4$ | 0.17x10$^4$ |
| 4.0 | 0.52x10$^4$ | 0.43x10$^4$ | 0.31x10$^4$ | 0.19x10$^4$ |
| 4.5 | 0.56x10$^4$ | 0.45x10$^4$ | 0.33x10$^4$ | 0.21x10$^4$ |
| 5.0 | 0.60x10$^4$ | 0.47x10$^4$ | 0.35x10$^4$ | 0.23x10$^4$ |
| 5.5 | 0.63x10$^4$ | 0,52x10$^4$ | 0.37x10$^4$ | 0.25x10$^4$ |
| 6.0 | 0.65x10$^4$ | 0.54x10$^4$ | 0.39x10$^4$ | 0.27x10$^4$ |
| 6.5 | 0.67x10$^4$ | 0.56x10$^4$ | 0.41x10$^4$ | 0.29x10$^4$ |
| 7.0 | 0.69x10$^4$ | 0.58x10$^4$ | 0.43x10$^4$ | 0.31x10$^4$ |
| 7.5 | 0.70x10$^4$ | 0.60x10$^4$ | 0.45x10$^4$ | 0.33x10$^4$ |
| 8.0 | 0.72x10$^4$ | 0.62x10$^4$ | 0.47x10$^4$ | 0.35x10$^4$ |

In **table T2,** we again have reported the work done by Adachi etal group regarding vortex line density L(t) as a function of counter flow velocity at different temperature. The steady state satisfy the relation $L = \gamma^2 v^2{}_{es}$, where $\gamma$ is temperature dependent parameter obtained numerically. Here, they observed that square root of vortex line density increases as a function of counter flow velocity for each temperature. The increase is low for T=1.3K but large for T=2.1K.

**Table T2:** This table also shows the result of steady state vortex line density L as a function of counter-flow velocities $v_{es}$ (cm/s) for different temperature T, results are taken from the paper of Adachi etal.[51], the steady states always satisfy the relation $\mathrm{L} = \gamma\, \mathrm{v}^2{}_{es}$, where $\gamma$ is temperature dependent parameter obtained numerically, the results indicate how does classical turbulence behaves in super fluid $^4$He**.**

| $v_{es}$ (cm/s) | $L^{\frac{1}{2}}$ (cm) | | | |
|---|---|---|---|---|
| | T=1.3K | T=1.6K | T=1.9K | T=2.1K |
| 0.00 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.10 | 8.75 | 10.57 | 27.08 | 29.43 |
| 0.15 | 12.64 | 15.48 | 32.35 | 38.87 |
| 0.20 | 18.72 | 22.56 | 41.29 | 43.24 |
| 0.25 | 22.32 | 24.87 | 45.62 | 46.68 |
| 0.30 | 25.69 | 27.05 | 50.26 | 51.16 |
| 0.35 | 28.45 | 32.17 | 54.47 | 55.34 |
| 0.40 | 32.87 | 34.37 | 57.79 | 58.67 |
| 0.45 | 34.65 | 36.58 | 62.23 | 63.44 |
| 0.50 | 38.23 | 39.09 | 66.45 | 67.56 |
| 0.55 | 41.54 | 43.16 | 68.75 | 69.08 |
| 0.60 | 43.68 | 45.77 | 72.09 | 73.43 |
| 0.65 | 45.79 | 47.68 | 74.23 | 75.65 |
| 0.70 | 47.27 | 49.05 | 76.56 | 77.89 |
| 0.80 | 49.58 | 51.24 | 78.83 | 79.26 |
| 0.90 | 55.26 | 53.43 | 81.95 | 83.43 |
| 1.00 | 52.44 | 55.28 | 83.56 | 87.67 |

In **table T3**, we have shown the evaluated results of incompressible and compressible kinetic energy $\mathrm{E}^{i}{}_{kin}$ and $E^{c}{}_{kin}$ (and the sum is $E = \mathrm{E}^{i}{}_{kin} + E^{c}{}_{kin}$) as a function of time t(s). Our theoretically obtained results show that both kinetic energies increase and decrease with time. However, the value of $E^{c}{}_{kin}$ is larger than $E^{c}{}_{kin}$. The fluctuation about their mean position value reveals the occurrence of exchange of interaction energy between all components of the total energy.

**Table T3:** An evaluated results of kinetic energy of swirls $E^{i}{}_{kin}$ (incompressible kinetic energy), $E^{c}{}_{kin}$ (compressible kinetic energy of sound waves) as a function of time t (s) for quantum turbulence (QT) in 2D BECs.

| t (sec) | $E^{i}{}_{kin}$ | $E^{c}{}_{kin}$ |
|---|---|---|
| 0.0 | 0.06 | 0.16 |
| 5.0 | 0.10 | 0.18 |
| 10.0 | 0.17 | 0.20 |
| 15.0 | 0.28 | 0.17 |
| 20.0 | 0.18 | 0.19 |
| 25.0 | 0.15 | 0.26 |
| 30.0 | 0.24 | 0.32 |
| 35.0 | 0.19 | 0.37 |
| 40.0 | 0.26 | 0.45 |
| 45.0 | 0.20 | 0.35 |
| 50.0 | 0.27 | 0.48 |
| 55.0 | 0.21 | 0.36 |
| 60.0 | 0.32 | 0.32 |
| 70.0 | 0.25 | 0.43 |
| 80.0 | 0.37 | 0.54 |
| 90.0 | 0.26 | 0.48 |
| 100.0 | 0.35 | 0.43 |

In **table T4,** we have shown the results of enstrophy as a function of time t (s) for 2D QT in non-uniform BECs. One defines the angle av.eraged enstrophy spectrum as $z(k) = (k/2)\int d\varphi |F(\omega)|^2$ where $F(\omega)$ is the Fourier transform in 2D space. The total enstropy is given by the integral $Z = \int_0^\infty z(k)dk$ where $z(k) = k^2 E^{i}{}_{kin}$ which is the well known result in the classical fluid dynamics. Our evaluated results of time evolution of enstrophy increase and decrease as observed in kinetic energy calculation. In **table T5,** we have shown the evaluated results of incompressible kinetic energy spectrum at t=100s for 2D non-uniform BECs. Here energy spectrum has been calculated as a function of wave vector k. Our obtained results show that the spectrum follows Kolmogorov's $k^{\frac{-5}{3}}$ law in the interval 2<k<4 and Saffman's $k^{-4}$ law for k>10. Between these two ranges there is a transitional zone 4<k<10, $E^{i}{}_{kin}$ exhibits no scaling behaviour. This also indicates that $E^{i}{}_{kin}$ depends on the geometry of the trapped BEC. In **table T6,** we have reported an experimental result of the emergence of QT in an atomic quantum fluids evidenced by the presence of vortex cores as well as vortex lines. These configurations are known as vortex tangles. We have shown the aspect ratio observed during time of flight (TOF) measurement in an oscillating BECs. In the table, normal BEC and turbulent BEC have been calculated as a function of TOF (ms). The results show that the aspect ratio in both cases increase with TOF (ms). This also confirms that the turbulent cloud does not expand like either quantum or thermal gas but keeps a fixed ratio.

**Table T4:** An evaluated result of enstrophy of 2D non-uniform Bose-Einstein condensate (BECs) as a function of time t(s), Enstrophy is calculated by the integral $Z = \int_0^\infty z(k)dk$ where $z(\mathrm{k}) = \mathrm{k}^2 E^i_{kin}$, which is well known result in fluid dynamics

| t(sec) | Enstrophy (Z) |
|---|---|
| 0.0 | 4,328 |
| 10 | 8.574 |
| 20 | 7.232 |
| 25 | 6.186 |
| 30 | 5.575 |
| 35 | 4.843 |
| 40 | 3.975 |
| 50 | 2,845 |
| 55 | 3.328 |
| 60 | 4.167 |
| 70 | 4.859 |
| 80 | 5.072 |
| 85 | 5.239 |
| 90 | 5.674 |
| 100 | 5.896 |

**Table T5:** An evaluated result of incompressible kinetic energy $E^i_{kin}$ as a function of wave-vector k at t=100s foe 2D non-uniform BEC

| K | Energy Spectrum |
|---|---|
| 10 | $6\text{x}10^{-7}$ |
| 15 | $14\text{x}10^{-7}$ |
| 20 | $24\text{x}10^{-8}$ |
| 25 | $18\text{x}10^{-8}$ |
| 30 | $12\text{x}10^{-8}$ |
| 35 | $9\text{x}10^{-8}$ |
| 40 | $13\text{x}10^{-9}$ |
| 45 | $8\text{x}10^{-9}$ |
| 50 | $4\text{x}10^{-9}$ |
| 55 | $21\text{x}10^{-10}$ |
| 60 | $7\text{x}10^{-10}$ |
| 65 | $57\text{x}10^{-11}$ |
| 70 | $42\text{x}10^{-11}$ |
| 80 | $7\text{x}10^{-11}$ |
| 90 | $53\text{x}10^{-12}$ |
| 100 | $40\text{x}10^{-12}$ |
| 120 | $32\text{x}10^{-12}$ |

**TableT6:**An experimental result of the aspect ratio of the normal BEC and the quantum turbulence in the case of oscillating BEC, results are taken during the flight of measurement (TOF) in an oscillating BEC[14]

| TOF(ms) | Normal BEC | Turbulent BEC |
|---|---|---|
| 4 | 0.817 | 0.627 |
| 6 | 0.925 | 0.708 |
| 8 | 0.989 | 0.729 |
| 10 | 1.047 | 0.764 |
| 12 | 1.098 | 0.793 |
| 14 | 1.124 | 0.815 |
| 16 | 1.167 | 0.827 |
| 18 | 1.248 | 0.835 |
| 20 | 1.269 | 0.846 |
| 22 | 1.326 | 0.850 |
| 24 | 1.345 | 0.861 |
| 26 | 1.384 | 0.874 |
| 28 | 1.405 | 0.882 |
| 30 | 1.428 | 0.896 |
| 32 | 1.447 | 0.909 |

In **table T7**, we have shown the evaluated results of spin dependent interaction energy as a function of $ka_h$. In fact the term $\frac{|E_s(k)|}{Nc_o n_0 \xi}$ has been calculated as a function of $ka_h$, Here, $N, c_0, n_0, \xi$ are total no of particles, spin-independent interaction coefficient, no of particle in the lowest state and the coherence length respectively. $E_s(k)$ is the spin-dependent interaction energy. Our obtained result show that the term decrease with $ka_h$. This results also indicate that the spectrum of spin-dependent interaction energy obey a -7/3 power law. In table T8, we have shown an evaluated result of time development of the exponent $n_p$ of the spectrum of spin dependent interaction energy follows a -7/3 power law. This power law has been derived by the scaling analysis of time development equation of the spin density vector[53]. This type of the behaviour has been noticed in the spin glass system. Spin glasses are magnetic system in which the interactions between the magnetic moments are in conflict with each other[54]. But spin glass in spinor BECs are different from magnetic spin glass due to the sense because the systems have no long –range order and spins are aligned in random direction. This result is the signature of spin quantum turbulence in spinor BECs. In **table T9,** we have shown the evaluated results of time dependent of q(t) an order parameter of spin glass system. For spinor BECs, this parameter also grows with the -7/3 power law. The result is in accordance with the behaviour of exponent $n_p$ which also approaches -7/3 power law behaviour, There is some recent results[55-68] which also reveals the similar behaviour.

**Table T7:**An evaluated result of spin dependent interaction energy at $t/\tau$ =4000 with characteristic time $\tau$, Here, the term $\frac{|E_s(k)|}{Nc_0 n_0 \xi}$ has been calculated as a function of $ka_h$. Here, the terms $E_s(k), n_0, c_0, \xi, a_h$ are spin dependent interaction energy, lowest state density, spin dependent interaction coefficient, coherence length and the characteristic length scale of spinor BEC. Result is the signature of the spin quantum turbulence (SQT) in atomic BEC

| $ka_h$ | $\frac{\lvert E_s(k)\rvert}{Nc_0 n_0 \xi}$ |
|---|---|
| 0.01 | $1234\text{x}10^{-3}$ |
| 0.02 | $768\text{x}10^{-3}$ |
| 0.05 | $532\text{x}10^{-3}$ |
| 0.10 | $276\text{x}10^{-3}$ |
| 0.15 | $59\text{x}10^{-3}$ |
| 0.20 | $33\text{x}10^{-3}$ |
| 0.30 | $10\text{x}10^{-3}$ |
| 0.50 | $87\text{x}10^{-4}$ |
| 1.00 | $56\text{x}10^{-4}$ |
| 5.00 | $23\text{x}10^{-4}$ |
| 10.00 | $87\text{x}10^{-5}$ |
| 20.00 | $54\text{x}10^{-5}$ |
| 30.00 | $35\text{x}10^{-5}$ |
| 50.00 | $12\text{x}10^{-5}$ |
| 60.00 | $8\text{x}10^{-5}$ |
| 70.00 | $5\text{x}10^{-5}$ |
| 80.00 | $3\text{x}10^{-5}$ |
| 90.00 | $2\text{x}10^{-5}$ |
| 100.00 | $9\text{x}10^{-6}$ |

**Table T8:**An evaluated result of time development of the exponent $n_p$ in the spectrum of spin dependent interaction energy, here the term $n_p$ is evaluated as a function of $t/\tau$, $\tau$ is the characteristic time, results show how spin turbulence behave in spinor BEC

| $t/\tau$ | $n_p$ (Power exponent) |
|---|---|
| 0.000 | -3.985 |
| 100 | -3.556 |
| 500 | -3.434 |
| 1000 | -3.267 |

| | |
|---|---|
| 1500 | -3.152 |
| 2000 | -3.086 |
| 2500 | -2.987 |
| 3000 | -2.846 |
| 3500 | -2.803 |
| 3800 | -2.794 |
| 4000 | -2.775 |
| 4200 | -2.764 |
| 4400 | -2.758 |
| 4500 | -2.745 |
| 4600 | -2.737 |
| 4800 | -2.678 |
| 5000 | -2.645 |

**Table T9:**An evaluated result of time dependence of the spin-glass order parameter $\sqrt{q(t)}$ , results are the signature of spin-glass behaviour of spin quantum turbulence in spinor BEC

| $t/\tau$ | $\sqrt{q(t)}$ |
|---|---|
| 0.00 | -2.164 |
| 100 | -2.006 |
| 500 | -1.983 |
| 1000 | -1.768 |
| 1500 | -1.674 |
| 2000 | -1.587 |
| 2500 | -1.534 |
| 3000 | -1.496 |
| 3200 | -1.408 |
| 3400 | -1.387 |
| 3600 | -1.324 |
| 3800 | -1.299 |
| 4000 | -1.276 |
| 4200 | -1.248 |
| 4400 | -1,205 |
| 4600 | -1-187 |
| 4800 | -1.162 |
| 5000 | -1.158 |

## CONCLUSION

From the above theoretical investigations and analysis, we have come across the following conclusions:

1. we have theoretically studied hydrodynamic instability and turbulence in quantum fluids. We have reviewed the work done in classical turbulence (CT) and quantum turbulence (QT) for the last 25 years. We have outlined the QT in varieties of quantum fluids including atomic BEC, 2D non-uniform BEC, oscillating BEC and spinor BEC respectively.
2. It has been observed that the classical turbulence (CT) in the case of super fluid helium can be well studied with vortex filament model and with the use of Biot-Savart law
3. Our evaluated results of time development of incompressible kinetic energy $E^{i}_{kin}$ and compressible kinetic energy $E^{c}_{kin}$ decrease and increase with time. These kinetic energies represent the kinetic energy of swirls and sound waves in the super flow and the total energy is the sum of these two energies. We have also observed that $E^{c}_{kin} > E^{i}_{kin}$. This result also indicates that both these energies stay almost stationary when the system reaches full turbulence. This has been observed in the case of 2D non uniform BECs. The fluctuation about their mean value reveals the occurrence of exchange of interaction energy between all components of the total energy.
4. Our evaluated results of enstrophy Z as a function of time t (s) also indicate decrease and increase behaviour. This is because the motion of vertex cause $E^{c}_{kin}$ and Z to increase and $E^{i}_{kin}$ to decrease. Enstrophy z(k) is defined as z(k)=k$^2$ $E^{i}_{kin}$ and this is a well known result in classical fluid dynamics.
5. Our evaluated results of incompressible kinetic energy spectrum at t=100s takes a power law for two ranges of spectral scaling. For the range 2<k<4, $E^{i}_{kin} \sim k^{\frac{-5}{3}}$ which is consistent with Kolmogorov $k^{\frac{-5}{3}}$ power law. For k>10, it follows k$^{-4}$ power law suggested by Soffman. Between these two ranges of spectral scaling, there is a transitional zone where $E^{i}_{kin}$(k) exhibit no scaling behaviour. It was also observed that in these two ranges, Z(k) follows $k^{\frac{-1}{3}}$ and $k^{-2}$ power law behaviour respectively.
6. We have reported an emergence of quantum turbulence kin an oscillatory BECS. Here, the aspect ratio of the normal BEC cloud and turbulent BEC cloud has been evaluated as a function of TOF(ms). The results show that the both the normal BEC and turbulent BEC increase with TOF(ms). It is also observed that turbulent cloud does not expand either like a quantum or thermal gas but keeps a fixed ratio.
7. Our evaluated results of spin dependent interaction energy obeys a -7/3 power law behaviour in the case of spinor BECs. Similarly the time development of exponent $n_p$ also obeys a -7/3 power law behaviour. This type of behaviour has been noticed in the case of magnetic spin –glass system. Now, here is the spin-glass behaviour in spinor BECs which is different from magnetic in the sense that it has no long range order and also the spins are randomly oriented. This has been observed by computing the time development of the spin-glass order parameter q(t) which grows as the spin turbulence develops a -7/3 power law.

8. The paper reviews the recent developments in the field of classical turbulence (CT) in super fluid helium and quantum turbulence (QT) in atomic BECs, 2D non-uniform BECs, oscillating BECs and spinor BECs. The evaluation has been made on the basis of quantum hydrodynamics. It has been found that classical turbulence(CT) can be well studied with the help of vortex filament model with the use of Biot-Savart law. The physics of quantum turbulence can be studied with the help of two coupled GP equation and scaling analysis. The material given in the paper can be quite useful in order to understand the fundamental physics of both CT and QT observed in quantum fluids.

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