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Research Article

Thermoelastic Surface Waves Propagation in Non-Homogeneous Viscoelastic Media of Higher Order

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Abstract: The propagation of surface waves in a non-homogeneous, isotropic, thermo-viscoelastic medium of higher order including time rate of strain has been discussed in this study. The problem has been solved analytically using linear inhomogeneity and the exact solution of velocity equations has been obtained for Stoneley, Rayleigh and Love type surface wave. In fact, these equations are in agreement with the corresponding classical results when the medium is free from viscosity, temperature and inhomogeneity.

Keywords: Inhomogeneous media, Variable density, Surface waves, Voigt equation

INTRODUCTION

The formulations and solutions of many problems of linear wave-propagation for homogeneous media are available in the literature of continuum mechanics of solids. In recent years, however, sufficient interest has arisen in the problem connected with bodies whose mechanical properties are functions of space, i.e. non-homogeneous bodies. This interest is mainly due to the advent of solid rocket propellants, polymeric materials and growing demand of engineering and industrial applications.

The dynamic theory of thermo-viscoelasticity is the study of dynamical interaction between thermal and mechanical fields in solid bodies and is of great importance in various fields of engineering such as earthquake engineering, aeronautics, soil engineering etc. Since our earth is a spherical body having finite dimension and the elastic waves generated must receive the effect of the boundaries. Naturally, this phenomenon leads us to the investigation of boundary waves or surface waves, i.e. the waves, which are confined to some surface during their propagation. As a result, Stoneley[1], Bullen[2], Ewing *et al*.[3], Hunters[4] Jones[5]and Jeffreys[6] have developed the theory of surface waves.

The effect of gravity on wave propagation in an elastic solid medium was first considered by Bromwich[7], treating the force of gravity as a type of body force. Love[8] extended the work of Bromwich [7]investigated the influence of gravity on super-facial waves and showed that the Rayleigh wave velocity is affected by the gravity field. Sezawa [9] studied the dispersion of elastic waves propagated on curved surfaces. Kakar *et al*.[10] studied the surface wave propagation in non homogeneous, general magneto-thermo, viscoelastic media.

Haskell[11] studied the dispersion of surface waves in multilayered media. A source on elastic waves is the monograph of Ewing *et al*.[12]. Biot [12] studied the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of hydrostatic nature and the medium to be incompressible. De and Sengupta[13] studied many problems of elastic waves and vibrations under the influence of gravity field. Sengupta and Acharya [14] studied the influence of gravity on the propagation of waves in a magnetoelastic layer. Brunelle [15] studied the surface wave propagation under initial tension of compression. Lamb[16] discussed the waves in elastic plate. Wave propagation in a thin two-layered laminated medium with stress couples under initial stresses was studied by Roy [17]. Datta[18] studied the effect of gravity on Rayleigh wave propagation in a homogeneous, isotropic elastic solid medium. Goda[19] studied the effect of non-homogeneity and anisotropy on Stoneley waves. The details are found in the work of Eringen and Sahubi[20]. Sharma and Kaur[21] studied Rayleigh waves in rotating thermoelastic solids with voids. Chattopadhyay et al.[22, 23] studied the propagation of G-type seismic waves in viscoelastic medium and they also discussed the effect of point source and heterogeneity on the propagation of SH- waves. Abd-Alla and Ahmed[24] studied the Rayleigh waves in an orthotropic magneto-elastic medium under gravity field and initial stress. Paria[25] discussed love waves in granular medium. Recently, Kakar *et al*.[26] analyzed a five-parameter Viscoelastic model under Dynamic Loading.

In this work, the problem of n^{th} order viscoelastic surface waves involving time rate of strain, the medium being isotropic and non-homogeneous has been studied under the influence of temperature. Biot's theory of incremental deformations has been used to obtain the wave velocity equation for Stoneley, Rayleigh and Love waves. Further, these equations are in complete agreement with the corresponding classical results in the absence of viscosity and thermal field, non-homogeneity of the material medium.

FORMULATION OF THE PROBLEM

Let M_1 and M_2 be two non-homogeneous, viscoelastic, isotropic, semi-finite media. They are perfectly welded in contact to prevent any relative motion or sliding before and after the disturbances and that the continuity of displacement, stress etc. hold good across the common boundary surface. Further the mechanical properties of M_1 are different from those of M_2. These media extend to an infinite great distance from the origin and are separated by a plane horizontal boundary and M_2 is to be taken above M_1.

Let Oxyz be a set of orthogonal Cartesian co-ordinates and let O be the any point on the plane boundary and Oz point vertically downward to the medium M_1. We consider the possibility of a type of wave traveling in the direction Ox, in such a manner that the disturbance is largely confined to the neighborhood of the boundary which implies that wave is a surface wave.

It is assume that at any instant, all particles in any line parallel to Oy having equal displacement and all partial derivatives with respect to y are zero. Further let us assume that u, v, w is the components of displacements at any point (x, y, z) at any time t.

The dynamical equations of motion for three-dimensional non-homogeneous, isotropic, viscoelastic solid medium in Cartesian co-ordinates are

$$\frac{\partial \tau_{11}}{\partial x}+\frac{\partial \tau_{12}}{\partial y}+\frac{\partial \tau_{13}}{\partial z}=\rho \frac{\partial^2 u}{\partial t^2}, \tag{1a}$$

$$\frac{\partial \tau_{12}}{\partial x}+\frac{\partial \tau_{22}}{\partial y}+\frac{\partial \tau_{23}}{\partial z}=\rho \frac{\partial^2 v}{\partial t^2}, \tag{1b}$$

$$\frac{\partial \tau_{13}}{\partial x}+\frac{\partial \tau_{23}}{\partial y}+\frac{\partial \tau_{33}}{\partial z}=\rho \frac{\partial^2 w}{\partial t^2}, \tag{1c}$$

Where ρ be the density of the material medium and $\tau_{ij} = \tau_{ji}\ \forall$ i, j are the stress components.

It is assumed that prior to the existence of any disturbance both the media are everywhere at the constant absolute temperature T_0.

The stress-strain relations for general isotropic, thermo, viscoelastic medium, are

$$\tau_{ij}= 2D_\mu e_{ij} + (D_\lambda \Delta - D_\beta T)\delta_{ij} \tag{2}$$

Where,

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \tag{3}$$ D_λ, D_μ, D_β are elastic constants

Introducing eq. (2) in eqs (1a), (1b), (1c), we get

$$D_\lambda \frac{\partial \Delta}{\partial x}+\Delta\frac{\partial D_\lambda}{\partial x}+2D_\mu\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial D_\mu}{\partial x}-D_\beta\frac{\partial T}{\partial x}+D_\mu\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]=\rho\frac{\partial^2 u}{\partial t^2}, \tag{4a}$$

$$D_\mu \nabla^2 v+\frac{\partial v}{\partial x}\frac{\partial D_\mu}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial D_\mu}{\partial z}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{4b}$$

$$\left\{\begin{array}{l} D_\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial D_\mu}{\partial x}+2D_\mu\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial w}{\partial z}\frac{\partial D_\mu}{\partial z}+D_\lambda\frac{\partial \Delta}{\partial z}+\Delta\frac{\partial D_\lambda}{\partial z}-D_\beta\frac{\partial T}{\partial z}-T\frac{\partial D_\beta}{\partial z}\end{array}\right\}=\rho\frac{\partial^2 w}{\partial t^2}. \tag{4c}$$

We assume that the non-homogeneities for the media M_1 and M_2 are given by

$$D_\lambda=\sum_{K=0}^{n}\lambda_K e^{mz}\frac{\partial^K}{\partial t^K},\ D_\mu=\sum_{K=0}^{n}\mu_K e^{mz}\frac{\partial^K}{\partial t^K},\ D_\beta=\sum_{K=0}^{n}\beta_K e^{mz}\frac{\partial^K}{\partial t^K},\ ,\rho=\rho_0 e^{mz} \tag{5a}$$

$$D'_\lambda=\sum_{K=0}^{n}\lambda'_K e^{lz}\frac{\partial^K}{\partial t^K},\ D_\mu=\sum_{K=0}^{n}\mu'_K e^{lz}\frac{\partial^K}{\partial t^K},\ D_\beta=\sum_{K=0}^{n}\beta_K e^{lz}\frac{\partial^K}{\partial t^K},\ ,\rho'=\rho'_0 e^{lz} \tag{5b}$$

Where λ_0, M_0, λ'_0, μ'_0 are elastic constants, whereas β_0, $\beta'_0$are thermal parameters are ρ_0, ρ'_0, m, n are constants. λ_K, μ_K (K = 0, 1, 2 ... n) are the parameters associated with K^{th} order viscoelasticity and β_K (K = 1, 2... n) are the thermal associated with K^{th} order. T is the absolute temperature over the initial temperature T_0. Due to temperature rise of the material medium, it has been observed that all the parameters representing elastic property, the effect of viscosity and thermal field depends on the temperature and ultimately depends on time t. In a thermo viscoelastic solid, the thermal parameters β_K (K = 0, 1 ... n) are given by

$\beta_K = (3\lambda_K + 2\mu_K)\,\alpha_t$, where α_t be the coefficient of linear expansion of solid.

$$\left(G_\lambda + G_\mu\right)\frac{\partial\Delta}{\partial x} + G_\mu \nabla^2 u + mG_\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) - G_\beta\frac{\partial T}{\partial x} = \rho_0\frac{\partial^2 u}{\partial t^2}\ , \tag{6a}$$

$$G_\mu \nabla^2 v + mG_\mu \frac{\partial v}{\partial z} = \rho_0\frac{\partial^2 v}{\partial t^2}\ , \tag{6b}$$

$$(G_\lambda + G_\mu)\frac{\partial\Delta}{\partial z} + G_\mu\nabla^2 w + \Delta G_\lambda m + 2G_\mu m\frac{\partial w}{\partial z} - G_\beta\frac{\partial T}{\partial z} - mG_B T = \rho_0\frac{\partial^2 w}{\partial t^2}. \tag{6c}$$

Where,

$$G_\lambda = \sum_{K=0}^{n}\lambda_K\frac{\partial^K}{\partial t^K},\quad G_\mu = \sum_{K=0}^{n}\mu_K\frac{\partial^K}{\partial t^K},\quad G_\beta = \sum_{K=0}^{n}\beta_K\frac{\partial^K}{\partial t^K},\quad \nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}. \tag{7}$$

To investigate the surface wave propagation along the direction of Ox, we introduce displacement potential ϕ (x, z, t) and ψ (x, z, t) which are related to the displacement components as follows:

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z},\quad w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}. \tag{8}$$

Substituting eq. (8) in eqs (6a), (6b) and (6c), we get

$$G_R\nabla^2\phi + mG_S\left(2\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial x}\right) - G_L T - = \frac{\partial^2\phi}{\partial t^2}, \tag{9a}$$

$$G_S\nabla^2 v + mG_S\frac{\partial v}{\partial z} = \frac{\partial^2 v}{\partial t^2}, \tag{9b}$$

$$G_S\nabla^2\psi + mG_P\frac{\partial\phi}{\partial x} + 2m\,G_S\frac{\partial\psi}{\partial z} = \frac{\partial^2\psi}{\partial t^2}. \tag{9c}$$

Where,

$$U_{KR}^2 = \frac{\lambda_K + 2\mu_K}{\rho_0},\ U_{KS}^2 = \frac{\mu_K}{\rho_0},\ U_{KP}^2 = \frac{\lambda_K}{\rho_0},\ U_{KL}^2 = \frac{\beta_K}{\rho_0} \tag{10a}$$

$$G_R = \sum_{K=0}^{n}U_{KR}^2\frac{\partial^K}{\partial t^K},\ G_S = \sum_{K=0}^{n}U_{KS}^2\frac{\partial^K}{\partial t^K},\ G_P = \sum_{K=0}^{n}U_{KP}^2\frac{\partial^K}{\partial t^K},\ G_L = \sum_{K=0}^{n}U_{KL}^2\frac{\partial^K}{\partial t^K} \tag{10b}$$

To determine T, Fourier's law of heat conduction

$$p\nabla^2 T = C_v\frac{\partial T}{\partial t} + T_0 G_L\frac{\partial}{\partial t}\left(\nabla^2\phi\right), \tag{11}$$

Where K be the thermal conductivity and obeys the law as given by $K = K_0 e^{mz}$, $p = \frac{K_0}{\rho_0}$ and C_v be the specific heat of the body at constant volume.

Further, similar relations in medium M_2 can be found out by replacing λ_K, μ_K, β_K, ρ_0 by λ'_K, μ'_K, β'_K, ρ'_0 and so on.

SOLUTION OF THE PROBLEM

Now our main objective to solve eqs (9a), (9b), (9c) and eq. (11)

For this, we seek the solutions in the following forms.

$$(\phi, \psi, T, v) = [f(z), g(z), T_1(z), h(z)]\, e^{i\alpha(x - ct)} \tag{12}$$

Using eq. (12) in eqs (9a), (9b), (9c) and eq. (11), we get a set of differential equations for the medium M_1 as follows:

$$\frac{d^2 f}{dz^2} + 2mf_1^2 \frac{df}{dz} + h_1^2 f + i\alpha m f_1^2 g - g_1^2 T_1 = 0, \tag{13a}$$

$$\frac{d^2 h}{dz^2} + m\frac{dh}{dz} + K_1^2 h = 0, \tag{13b}$$

$$\frac{d^2 g}{dz^2} + 2m\frac{dg}{dz} + K_1^2 g + i\alpha m l_1^2 f = 0, \tag{13c}$$

$$\frac{d^2 T_1}{dz^2} + AT_1 + B\left(\frac{d^2 f}{dz^2} - \alpha^2 f\right) = 0, \tag{13d}$$

Where,

$$f_1^2 = \frac{\sum_{K=0}^{n} U_{KS}^2 (-i\alpha c)^K}{\sum_{K=0}^{n} U_{KR}^2 (-i\alpha c)^K},\ h12 = \frac{\alpha^2 c^2}{\sum_{K=0}^{n} U_{KR}^2 (-i\alpha c)^K} - \alpha^2,\ K_1^2 = \frac{\alpha^2 c^2}{\sum_{K=0}^{n} U_{KS}^2 (-i\alpha c)^K} - \alpha^2, \tag{14a}$$

$$l_1^2 = \frac{\sum_{K=0}^{n} U_{KP}^2 (-i\alpha c)^K}{\sum_{K=0}^{n} U_{KS}^2 (-i\alpha c)^K},\ g12 = \frac{\sum_{K=0}^{n} U_{KL}^2 (-i\alpha c)^K}{\sum_{K=0}^{n} U_{KR}^2 (-i\alpha c)^K},\ A = \frac{C_v\, i\alpha c}{p} - \alpha^2,\ B = \frac{i\alpha c T_0}{p} G_L \tag{14b}$$

and those for the medium M_2 are given by

$$\frac{d^2 f}{dz^2} + 2lf_1'^2 \frac{df}{dz} + h_1'^2 f + i\alpha l f_1'^2 g - g_1'^2 T_1 = 0, \tag{15a}$$

$$\frac{d^2 h}{dz^2} + l\frac{dh}{dz} + K_1'^2 h = 0, \tag{15b}$$

$$\frac{d^2 g}{dz^2} + 2l\frac{dg}{dz} + K_1'^2 g + i\alpha l . l_1'^2 f = 0, \tag{15c}$$

$$\frac{d^2 T_1}{dz^2} + A'T_1 + B'\left(\frac{d^2 f}{dz^2} - \alpha^2 f\right) = 0. \tag{15d}$$

Where,

$$f_1'^2 = \frac{\sum_{K=0}^{n} U_{KS}'^2 (-i\alpha c)^K}{\sum_{K=0}^{n} U_{KR}'^2 (-i\alpha c)^K},\ h_1'^2 = \frac{\alpha^2 c^2}{\sum_{K=0}^{n} U_{KR}'^2 (-i\alpha c)^K} - \alpha^2,\ K_1'^2 = \frac{\alpha^2 c^2}{\sum_{K=0}^{n} U_{KS}'^2 (-i\alpha c)^K} - \alpha^2, \tag{16a}$$

$$l_1'^2 = \frac{\sum_{K=0}^{n} U'^2_{KP}\,(-i\alpha c)^K}{\sum_{K=0}^{n} U'^2_{KS}\,(-i\alpha c)^K},\quad g_1'^2 = \frac{\sum_{K=0}^{n} U'^2_{KL}\,(-i\alpha c)^K}{\sum_{K=0}^{n} U'^2_{KR}\,(-i\alpha c)^K},\quad B' = \frac{i\alpha c T_0}{p'}\,G'_L \tag{16b}$$

Eqs (13) and (15) must have exponential solutions in order that f, g, T_1, h will describe surface waves, and they must become varnishing small as $z \to \infty$.

Hence for the medium M_1

$$\phi\,(x, z, t) = \left\{A_1 e^{-\lambda_1 z} + B_1 e^{-\lambda_2 z} + C_1 e^{-\lambda_3 z}\right\} e^{i\alpha(x-ct)}$$

$$\psi\,(x, z, t) = \left\{A_2 e^{-\lambda_1 z} + B_2 e^{-\lambda_2 z} + C_2 e^{-\lambda_3 z}\right\} e^{i\alpha(x-ct)}$$

$$T\,(x, z, t) = \left\{A_3 e^{-\lambda_1 z} + B_3 e^{-\lambda_2 z} + C_3 e^{-\lambda_3 z}\right\} e^{i\alpha(x-ct)} \tag{17a}$$

$$v\,(x, z, t) = C e^{-\lambda_4 z + i\alpha(x-ct)}$$

And similarly for the medium M_2 are given by

$$\phi\,(x, z, t) = \left\{A'_1 e^{-\lambda'_1 z} + B'_1 e^{-\lambda'_2 z} + C'_1 e^{-\lambda'_3 z}\right\} e^{i\alpha(x-ct)}$$

$$\psi\,(x, z, t) = \left\{A'_2 e^{-\lambda'_1 z} + B'_2 e^{-\lambda'_2 z} + C'_2 e^{-\lambda'_3 z}\right\} e^{i\alpha(x-ct)}$$

$$T\,(x, z, t) = \left\{A'_3 e^{-\lambda'_1 z} + B'_3 e^{-\lambda'_2 z} + C'_3 e^{-\lambda'_3 z}\right\} e^{i\alpha(x-ct)} \tag{17b}$$

$$v\,(x, z, t) = C' e^{-\lambda'_4 z + i\alpha(x-ct)}$$

Where λ_j and λ'_j (j = 1, 2, 3) are the real roots of the eqs

$$\lambda^6 + \xi_1\lambda^5 + \xi_2\lambda^4 + \xi_3\lambda^3 + \xi_4\lambda^2 + \xi_5\lambda + \xi_6 = 0, \tag{18}$$

Where,

$$\xi_1 = 2m\,\{1 + f_1^2\},\ \xi_2 = K_1^2 + A + 4m^2 + h_1^2 + Bg_1^2, \tag{19a}$$

$$\xi_3 = 2mA + 2f_1^2\, m\,(K_1^2 + A) + 2mh_1^2 + 2mBg_1^2 \tag{19b}$$

$$\xi_4 = AK_1^2 + 4m^2A\, f_1^2 + (K_1^2 + A)\, h_1^2 + \alpha^2\, m^2 l_1^2\, f_1^2 + BK_1^2\, g_1^2 - \alpha^2 Bg_1^2, \tag{19c}$$

$$\xi_5 = 2mAK_1^2\, f_1^2 + 2mAh_1^2 - 2m\,\alpha^2 Bg_1^2, \tag{19d}$$

$$\xi_6 = AK_1^2\, h_1^2 + A\alpha^2\, m^2\, l_l^2\, f_1^2 - \alpha^2 B\, K_1^2\, g_1^2. \tag{19e}$$

$$\lambda'^6 + \xi'_1\lambda'^5 + \xi'_2\lambda'^4 + \xi'_3\lambda'^3 + \xi'_4\lambda'^2 + \xi'_5\lambda' + \xi'_6 = 0 \tag{20}$$

Where,

$$\xi'_1 = 2l\,\{1 + f_1'^2\},\ \xi'_2 = K_1'^2 + A' + 4l^2 + h_1'^2 + B'g_1'^2, \tag{21a}$$

$$\xi'_3 = 2lA' + 2lf_1'^2\,(K_1'^2 + A) + 2lh_1'^2 + 2l\,B'g_1'^2, \tag{21b}$$

$\xi'_4 = A'K_1'^2 + 4l^2A'\, f_1'^2 + (K_1'^2 + A')\, h_1'^2 + \alpha^2 l^2 l_1'^2\, f_1'^2 + B'\, K_1'^2\, g_1'^2 - \alpha^2\, B'\, g_1'^2,$ (21c)

$\xi'_5 = 2lA'K_1'^2\, f_1'^2 + 2lA'h_1'^2 - 2l\alpha^2 B'g_1'^2,$ (21d)

$\xi'_6 = A'K_1'^2\, h_1'^2 + A'\alpha^2 l^2\, l_l'^2\, f_1'^2 - \alpha^2 B'\, K_1'^2\, g_1'^2.$ (21e)

$\lambda_4 = \{m + (m^2 - 4\, K_1^2)^{1/2}\}/\, 2,$ (21f)

$\lambda'_4 = \{l + (l^2 - 4\, K_1'^2)^{1/2}\}/\, 2.$ (21g)

Where the symbol used in eqs (19) and (21) are given by eqs (14) and (16).

The constants A_j, B_j, C_j (j = 1, 2, 3) are related with A'_j, B'_j, C'_j (j = 1, 2, 3) in eqs (17a) and (17b) by means of first equations in eqs (13) and (15).

Equating the coefficients of $e^{-\lambda_1 z}$, $e^{-\lambda_2 z}$, $e^{-\lambda_3 z}$, $e^{-\lambda'_1 z}$, $e^{-\lambda'_2 z}$, $e^{-\lambda'_3 z}$ to zero, after substituting eqs (17a) and (17b) in the first and 3rd equations of eq. (13) and eq. (15) respectively, we get

$A_2 = \gamma_1\, A_1,\ B_2 = \gamma_2\, B_1,\ C_2 = \gamma_3\, C_1,$ (22a)

$A_3 = \delta_1\, A_1,\ B_3 = \delta_2\, B_1,\ C_3 = \delta_3\, C_1,$ (22b)

Where,

$$\gamma_j = \frac{-i\alpha m l_1^2}{\lambda_j^2 - 2m\lambda_j + K_1^2}\ (j = 1, 2, 3), \quad (23a)$$

$$\delta_j = \frac{1}{g_1^2}\,[\lambda_j^2 - 2m\, f_1^2 \lambda_j + h_1^2 + i\,\alpha\, m\, f_1^2 \gamma_j]\ (j = 1, 2, 3). \quad (23b)$$

Similar result holds for medium M_2 and usual symbols replacing by dashes respectively.

BOUNDARY CONDITIONS

(i) The displacement components, temperature and temperature flux at the boundary surface between the media M_1 and M_2 must be continuous at all times and positions.

i.e. $\left[u,\, v,\, w,\, T,\ p\dfrac{\partial T}{\partial z}\right]_{M_1} = \left[u,\, v,\, w,\, T,\, p'\dfrac{\partial T}{\partial z}\right]_{M_2}$

(ii) The stress components τ_{31}, τ_{32}, τ_{33} must be continuous at the boundary z = 0.

i.e. $[\tau_{31},\, \tau_{32},\, \tau_{33}]_{M_1} = [\tau_{31},\, \tau_{32},\, \tau_{33}]_{M_2}$ at z = 0 respectively

Where,

$$\tau_{31} = D_\mu \left(2\,\frac{\partial^2\phi}{\partial x\,\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right), \quad (24a)$$

$$\tau_{32} = D_\mu\,\frac{\partial v}{\partial z}, \quad (24b)$$

$$\tau_{33} = D_\lambda \nabla^2 \phi + 2D_\mu \left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial x \partial z}\right) - D_B T. \quad (24c)$$

Applying the boundary conditions, we get

$$A_1 (1 - i\,\gamma_1\zeta_1) + B_1 (1 - i\,\gamma_2\zeta_2) + C_1 (1 - i\,\gamma_3\zeta_3) - A'_1 (1 - i\,\gamma'_1\zeta'_1) - B'_1 (1 - i\,\gamma'_2\zeta'_2) - C'_1 (1 - i\,\gamma'_3\zeta'_3) = 0 \quad (25a)$$

$$C = C' \quad (25b)$$

$$A_1 (\gamma_1 + i\zeta_1) + B_1 (\gamma_2 + i\zeta_2) + C_1 (\gamma_3 + i\zeta_3) - A'_1 (\gamma'_1 + i\zeta'_1) - B'_1 (\gamma'_2 + i\zeta'_2) - C'_1 (\gamma'_3 + i\zeta'_3) = 0 \quad (25c)$$

$$\delta_1 A_1 + \delta_2 B_1 + \delta_3 C_1 = \delta'_1 A'_1 + \delta'_2 B'_1 + \delta'_3 C'_1 \quad (25d)$$

$$p\lambda_1\delta_1 A_1 + p\lambda_2\delta_2 B_1 + p\lambda_3\delta_3 C_1 - p'\lambda'_1\delta'_1 A'_1 + p'\lambda'_2\delta'_2 B'_1 - p'\lambda'_3\delta'_3 C'_1 = 0 \quad (25e)$$

$$m^*_K [(2i\,\zeta_1 + \gamma_1 + \zeta_1^2\gamma_1) A_1 + (2i\,\zeta_2 + \gamma_2 + \zeta_2^2\gamma_2) B_1 + (2i\,\zeta_3 + \gamma_3 + \zeta_3^2\gamma_3) C_1] =$$

$$m'^*_K [(2i\,\zeta'_1 + \gamma'_1 + \zeta_1'^2\gamma'_1) A'_1 + (2i\,\zeta'_2 + \gamma'_2 + \zeta_2'^2\gamma'_2) B'_1 + (2i\,\zeta'_3 + \gamma'_3 + \zeta_3'^2\gamma'_3) C'_1] \quad (25f)$$

$$m^*_K [-\lambda_4 C] = m'^*_K [-\lambda'_4 C'] \quad (25g)$$

$$A_1 [(l^*_K + (\mu_e)^*_K H_0^2)(\zeta_1^2 - 1) + 2\,m^*_K (\zeta_1^2 - i\zeta_1) - b^*_K \delta_1] + B_1 [(l^*_K + (\mu_e)^*_K H_0^2)(\zeta_2^2 - 1)$$

$$+ 2\,m^*_K (\zeta_2^2 - i\zeta_2) - b^*_K \delta_2] + C_1 [(l^*_K + (\mu_e)^*_K H_0^2)(\zeta_3^2 - 1) + 2\,m^*_K (\zeta_3^2 - i\zeta_3) - b^*_K \delta_3] =$$

$$A'_1 [(l'^*_K + (\mu'_e)^*_K H_0^2)(\zeta_1'^2 - 1) + 2\,m'^*_K (\zeta_1'^2 - i\zeta'_1) - b'^*_K \delta'_1] + B'_1 [(l'^*_K + (\mu'_e)^*_K H_0^2)(\zeta_2'^2 - 1)$$

$$+ 2\,m'^*_K (\zeta_2'^2 - i\zeta'_2) - b'^*_K \delta'_2] + C'_1 [(l'^*_K + (\mu'_e)^*_K H_0^2)(\zeta_3'^2 - 1) + 2\,m'^*_K (\zeta_3'^2 - i\zeta'_3) - b'^*_K \delta'_3] \quad (25h)$$

where, $\zeta_j = \dfrac{\lambda_j}{\alpha}$, $\zeta'_j = \dfrac{\lambda'_j}{\alpha}$, $j = 1, 2, 3$

and

$$\lambda^*_K = \sum_{K=0}^{n} \lambda_K (-i\alpha c)^K, \quad m^*_K = \sum_{K=0}^{n} \mu_K (-i\alpha c)^K, \quad b^*_K = \sum_{K=0}^{n} \beta_K (-i\alpha c)^K,$$

$$l'^*_K = \sum_{K=0}^{n} \lambda'_K (-i\alpha c)^K, \quad m'^*_K = \sum_{K=0}^{n} \mu'_K (-i\alpha c)^K, \quad b'^*_K = \sum_{K=0}^{n} \beta'_K (-i\alpha c)^K,$$

From eqs (25b) and (25g), we have

$C = C' = 0$.

Thus there is no propagation of displacement v. Hence SH-waves do not occur in this case.

Finally, eliminating the constants A_1, B_1, C_1, A'_1, B'_1, C'_1, from the remaining equations, we get

$$\det(a_{ij}) = 0, \quad i, j = 1, 2, 3, 4, 5, 6. \quad (26)$$

Where,

$a_{11} = 1 - i\gamma_1\zeta_1$, $a_{12} = 1 - i\gamma_2\zeta_2$, $a_{13} = 1 - i\gamma_3\zeta_3$, $a_{14} = (i\,\gamma'_1\zeta'_1 - 1)$,

$a_{15} = (i\gamma'_2\zeta'_2 - 1)$, $a_{16} = (i\gamma'_3\zeta'_3 - 1)$,

$a_{21} = \gamma_1 + i\zeta_1$, $a_{22} = \gamma_2 + i\zeta_2$, $a_{23} = \gamma_3 + i\zeta_3$, $a_{24} = (\gamma'_1 + i\zeta'_1)$, $a_{25} = (\gamma'_2 + i\zeta'_2)$,

$a_{26} = (\gamma'_3 + i\zeta'_3)$,

$a_{31} = \delta_1$, $a_{32} = \delta_2$, $a_{33} = \delta_3$, $a_{34} = -\delta'_1$, $a_{35} = -\delta'_2$, $a_{36} = -\delta'_3$,

$a_{41} = p\lambda_1\delta_1$, $a_{42} = p\lambda_2\delta_2$, $a_{43} = p\lambda_3\delta_3$, $a_{44} = -p'\lambda'_1\delta'_1$, $a_{45} = -p'\lambda'_2\delta'_2$,

$a_{46} = -p'\lambda'_3\delta'_3$,

$a_{51} = m_K^*(2i\zeta_1 + \gamma_1 + \gamma_1\zeta_1^2)$, $a_{52} = m_K^*(2i\zeta_2 + \gamma_2 + \gamma_2\zeta_2^2)$,

$a_{53} = m_K^*(2i\zeta_3 + \gamma_3 + \gamma_3\zeta_3^2)$,

$a_{54} = m'^*_K(2i\zeta'_1 + \gamma'_1 + \gamma'_1\zeta_1'^2)$, $a_{55} = m'^*_K(2i\zeta'_2 + \gamma'_2 + \gamma'_2\zeta_2'^2)$,

$a_{56} = m'^*_K(2i\zeta'_3 + \gamma'_3 + \gamma'_3\zeta_3'^2)$,

$a_{61} = (l_K^*)(\zeta_1^2 - 1) + 2m_K^*(\zeta_1^2 - i\zeta_1) - b_K^*\delta_1$,

$a_{62} = (l_K^*)(\zeta_2^2 - 1) + 2m_K^*(\zeta_2^2 - i\zeta_2) - b_K^*\delta_2$,

$a_{63} = (l_K^*)(\zeta_3^2 - 1) + 2m_K^*(\zeta_3^2 - i\zeta_3) - b_K^*\delta_3$,

$a_{64} = (l'^*_K)(\zeta_1'^2 - 1) + 2m'^*_K(\zeta_1'^2 - i\zeta'_1) - b'^*_K\delta'_1$,

$a_{65} = (l'^*_K)(\zeta_2'^2 - 1) + 2m'^*_K(\zeta_2'^2 - i\zeta'_2) - b'^*_K\delta'_2$,

$a_{66} = (l'^*_K)(\zeta_3'^2 - 1) + 2m'^*_K(\zeta_3'^2 - i\zeta'_3) - b'^*_K\delta'_3$,

From eq. (26), we obtain velocity of surface waves in common boundary between two viscoelastic, non-homogeneous solid media under the influence of thermal field, where the viscosity is of general n^{th} order involving time rate of change of strain.

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Stoneley Waves: It is the generalized form of Rayleigh waves in which we assume that waves are propagated along the common boundary of the two semi-infinite media M_1 and M_2. Thus eq. (26) determine the wave velocity equation for Stoneley waves in the case of general thermo viscoelastic, non-homogeneous solid media of n^{th} order involving time rate of strain. Clearly, from eq. (26), the wave velocity equation for Stoneley waves depends upon the non-homogeneity of the material medium, temperature and viscous field follows. This equation, of course, is in well agreement with the corresponding classical result, when the effects of thermal, viscous field and non-homogeneity are absent.

Rayleigh Waves: To investigate the possibility of Rayleigh waves in a thermo viscoelastic, non-homogeneous elastic media, we replace media M_2 by vacuum, in the proceeding problem; we also note the SH-waves do not occur in this case.

Since the temperature difference across the boundary is always small so thermal condition given by

$$\frac{\partial T}{\partial z} + hT = 0 \text{ at } z = 0 \text{ respectively} \tag{27}$$

Thus eqs (25f) and (25h) reduces to,

$$(2i\,\zeta_1 + \gamma_1 + \gamma_1\zeta_1^2)\,A_1 + (2i\,\zeta_2 + \gamma_2 + \gamma_2\zeta_2^2)\,B_1 + (2i\,\zeta_3 + \gamma_3 + \gamma_3\zeta_3^2)\,C_1 = 0 \tag{28a}$$

$$[(l_K^*)\,(\zeta_1^2 - 1) + 2\,m_K^*\,(\zeta_1^2 - i\zeta_1) - b_K^*\,\delta_1]\,A_1 + [(l_K^*)\,(\zeta_2^2 - 1) + 2\,m_K^*\,(\zeta_2^2 - i\zeta_2) - b_K^*\,\delta_2]\,B_1$$

$$+ [(l_K^*)\,(\zeta_3^2 - 1) + 2\,m_K^*\,(\zeta_3^2 - i\zeta_3) - b_K^*\,\delta_3]\,C_1 = 0 \tag{28b}$$

From eq. (27), we have

$$(\lambda_1 - h)\,\delta_1\,A_1 + (\lambda_2 - h)\,\delta_2\,B_1 + (\lambda_3 - h)\,\delta_3\,C_1 = 0 \tag{28c}$$

Eliminating A_1, B_1 and C_1 from eqs (28a), (28b) and (28c), we get

$$\det(b_{ij}) = 0,\ i, j = 1, 2, 3. \tag{29}$$

Where,

$b_{11} = (2i\,\zeta_1 + \gamma_1 + \gamma_1\zeta_1^2)$, $b_{12} = (2i\,\zeta_2 + \gamma_2 + \gamma_2\zeta_2^2)$, $b_{13} = (2i\,\zeta_3 + \gamma_3 + \gamma_3\zeta_3^2)$,

$b_{21} = [(l_K^*)\,(\zeta_1^2 - 1) + 2\,m_K^*\,(\zeta_1^2 - i\zeta_1) - b_K^*\,\delta_1]$,

$b_{22} = [(l_K^*)\,(\zeta_2^2 - 1) + 2\,m_K^*\,(\zeta_2^2 - i\zeta_2) - b_K^*\,\delta_2]$,

$b_{23} = [(l_K^*)\,(\zeta_3^2 - 1) + 2\,m_K^*\,(\zeta_3^2 - i\zeta_3) - b_K^*\,\delta_3]$,

$b_{31} = (\lambda_1 - h)\,\delta_1$, $b_{32} = (\lambda_2 - h)\,\delta_2$, $b_{33} = (\lambda_3 - h)\,\delta_3$.

Thus eq. (29), gives the wave velocity equation for Rayleigh waves in a non-homogeneous, thermo viscoelastic solid media of n^{th} order involving time rate of strain.

From eq. (29), it is follows that dispersion equation of Rayleigh waves depends upon the non-homogeneity, the viscous and thermal fields.

This equation, of course, is in complete agreement with the corresponding classical result by Bullen [2], when the effects of thermal viscous field and non-homogeneity are absent.

Love Waves: To investigate the possibility of love waves in a non-homogeneous, viscoelastic solid media, we replace medium M_2 is obtained by two horizontal plane surfaces at a distance H-apart, while M_1 remains infinite.

For medium M_1, the displacement component ν remains same as in general case given by eq. (17).

For the medium M_2, we preserve the full solution, since the displacement component along y-axis i.e. v no longer diminishes with increasing distance from the boundary surface of two media.

$$\text{Thus } v' = C_1\, e^{\lambda'_4 z + i\alpha(x - ct)} + C_2\, e^{-\lambda'_4 z + i\alpha(x - ct)} \tag{30}$$

In this case, the boundary conditions are

(i) v and τ_{32} are continuous at z = 0

(ii) $\tau'_{32} = 0$ at $z = -H$.

Applying boundary conditions (i) and (ii) and using eqs (17) and (25), we get

$$C= C_1 + C_2 \tag{31}$$

$$-m^*_K \lambda_4 C= (\mu'_K)^* [\lambda'_4 C_1 - \lambda'_4 C_2] \tag{32}$$

$$C_1 e^{-\lambda'_4 H} - C_2 e^{\lambda'_4 H} = 0 \tag{33}$$

On eliminating the constants C, C_1 and C_2 from eqs (26), (27) and (28), we get

$$\tanh(\lambda'_4 H) = -\frac{\lambda_4 \mu^*_K}{\lambda'_4 (\mu'_K)^*}. \tag{34}$$

Thus eq. (34) gives the wave velocity equation for Love waves in a non-homogeneous, thermo viscoelastic solid medium of n^{th} order involving time rate of strain. Clearly it depends upon the non-homogeneity and viscous fields and independent of thermal field.

CONCLUSIONS

I. The present study reveals the effects of non-homogeneity, viscous and thermal fields in the wave velocity equations corresponding to Stoneley waves, Rayleigh waves and Love waves.

II. Further it is investigated that viscoelastic surface waves are affected by the time rate of strain parameters. These parameters influence the wave velocity to an extent depending on the corresponding constants characterizing the thermo and viscoelasticity of the material. So the results of this analysis become useful in circumstances where these effects cannot be neglected. These velocities depend upon the wave number 'α' confirming that these waves are affected by non-homogeneity of the material medium.

III. Also it is noted from eq. (34) is that Love waves does not depends on temperature; these are only affected by viscous and non-homogeneity of the material medium. In absence of all fields and non-homogeneity, the dispersion equation is in complete agreement with the corresponding classical result.

IV. It is noted that for Rayleigh waves in a non-homogeneous, general thermo viscoelastic solid medium of higher order including time rate of change of strain we find that the wave velocity equation proves that there is dispersion of waves due to the presence of non-homogeneity, temperature field and viscosity. The results are in complete agreement with the corresponding classical results in the absence of all fields and compression.

V. The wave velocity equation of Stoneley waves is very similar to the corresponding problem in the classical theory of elasticity. Here also there is dispersion of waves due to the presence of non-homogeneity and temperature and viscoelastic nature of the solid. Also wave velocity equation of this generalized type of surface waves in non-homogeneous thermo viscoelastic solid media of higher order including time rate of strain is in complete agreement with the corresponding classical result in the absence of all fields and non-homogeneity.

VI. The solution of wave velocity equation for Stoneley waves cannot be determined by easy analytical methods however, we can apply numerical techniques to solve this determinantal equation by choosing suitable values of physical constants for both media M_1 and M_2.

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