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Any *l*-state solutions of the Schrödinger equation for the Woods-Saxon potential in arbitrary dimensions

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Abstract: The exact solutions of the Woods-Saxon potential with centrifuge term in arbitrary dimensions *D* were obtained within the proper quantization rule. Dealing with the centrifuge term in Woods-Saxon potential by the Pekeris approximation. The inter-dimensional degeneracies for various orbital quantum number l and dimensional space D were studied.

Keywords: Woods-Saxon potential; Proper quantization rule; Pekeris approximation

INTRODUTION

Since the exact solution contains almost all quantum information, it played a very important role in quantum computing, but only a few simple quantum systems had exact solutions. Therefore, finding an exact solution was a significant and challenging task. At present, people have developed many effective methods for finding exact solutions, such as: factorization [1], supersymmetry [2], separation variables [3] and SWKB methods [4].

The Woods-Saxon potential [5] proposed by Roger D. Woods and David S. Saxon in the study of the elastic scattering between 20MeV protons and heavy nuclear. It was an important mean field atomic potential that was used to describe the interaction between a single nucleus and all other nucleuses. Woods-Saxon potential was widely used in atomic structures [6], atomic reactions [7], atomic scattering [8,9],

and particle physics[10]. Guo et al.[11] calculated the exact solution of the *s*-wave Dirac equation with spin and pseudospin symmetry single particles that moving in the central Woods-Saxon potential. Gönül et al.[12] calculated the exact solution of the Klein-Gordon equation for single neutron motion in the central Woods-Saxon potential.

However, the existence of the centrifugal term $l(l+1)/r^2$, Schrödinger equation, Dirac equation and Klein-Gordon equation were all limited to the *s*-wave (l=0) exact solution. Based on the above literature research, this paper solved the Schrödinger equation of the Woods-Saxon potential with a centrifugal term at any dimension $l(l\neq 0)$ state by the proper quantization rule method.

METHODS

The one-dimensional Schrödinger equation can be expressed as:

$$\frac{d^2}{dx^2}\psi(x)=-\frac{2M}{\hbar^2}\left[E-V(x)\right]\psi(x) \tag{1}$$

Equation (1) can be written as a nonlinear Riccati equation:

$$\frac{d\phi(x)}{dx}=-\frac{2M}{\hbar^2}\left[E-V(x)\right]-\phi(x)^2 \tag{2}$$

Where $\phi(x)=\psi(x)^{-1}d\psi(x)/dx$ was the logarithmic derivative of the wave function $\psi(x)$. When $E\geq V(x)$, $\phi(x)$ decreases with respect to x monotonically. In order to study the exact solution of the one-dimensional Schrödinger equation, Ma and Xu proposed an exact quantization rule[13-15]:

$$\int_{x_A}^{x_B}k(x)dx=N\pi+\int_{x_A}^{x_B}k'(x)\frac{\phi(x)}{\phi'(x)}dx \tag{3}$$

Where wave vector $k(x)=\sqrt{2M\left[E-V(x)\right]}/\hbar$, x_A and x_B were two turning points of $E=V(x)$. The $N=n+1$ was the number of nodal points of $\phi(x)$ in the range of $E\geq V(x)$ and n was the number of nodal points of $\psi(x)$. If N=1, the formula (3) can be expressed as:

$$\int_{x_{0A}}^{x_{0B}}k_0(x)dx=\pi+\int_{x_{0A}}^{x_{0B}}k_0'(x)\frac{\phi_0(x)}{\phi_0'(x)}dx,\quad k_0(x)=\sqrt{2M\left[E_0-V(x)\right]}/\hbar \tag{4}$$

The second item of the above formula can be expressed as:

$$\int_{x_{0A}}^{x_{0B}}k_0'(x)\frac{\phi_0(x)}{\phi_0'(x)}dx=\int_{x_{0A}}^{x_{0B}}k_0(x)dx-\pi \tag{5}$$

Substituting it into equation (3),we can obtain:

$$\int_{x_A}^{x_B}k(x)dx-\int_{x_{0A}}^{x_{0B}}k_0(x)dx=(N-1)\pi=n\pi \tag{6}$$

Equation (4) can be written in the same form:

$$\int_{r_A}^{r_B}k(r)dr-\int_{r_{0A}}^{r_{0B}}k_0(r)dr=n\pi \tag{7}$$

Equations (6) and (7) are called proper quantization rule[16-19].

The Woods-Saxon potential with D-dimensional l state can be expressed as (select the natural unit $\hbar^2=2M=1$ for simplicity):

$$V_{eff}(r)=-\frac{v_0}{1+e^{\frac{r-R_0}{a}}}+\left(l+\frac{D-1}{2}\right)\left(l+\frac{D-3}{2}\right)\frac{1}{r^2} \tag{8}$$

The second term in equation (8) was the centrifugal term, and it had an analytical solution when the centrifugal term was zero. The Pekeris method[12, 20] was used to approximate the centrifugal term and introduced a new variable $x=r-R_0$:

$$\begin{aligned}\frac{1}{r^2}&\approx\frac{1}{R_0^2}\left[D_0-D_1\frac{e^{-x/a}}{1+e^{-x/a}}+D_2\frac{e^{-2x/a}}{\left(1+e^{-x/a}\right)^2}\right]\\&=\frac{1}{R_0^2}\left[D_0-\frac{D_1}{1+e^{x/a}}+\frac{D_2}{\left(1+e^{x/a}\right)^2}\right]\end{aligned} \tag{9}$$

Where D_0, D_1 and D_2 were the coupling coefficients. Equation (9) was expanded at $x=0$, and the second order term was obtained:

$$\frac{1}{r^2}\approx\frac{1}{R_0^2}\left[1-2\left(\frac{x}{R_0}\right)+3\left(\frac{x}{R_0}\right)^2+\cdots\right] \tag{10}$$

Equation (9) can also be expanded at the minimum point $r\approx R_0$, and the coupling coefficient of the Woods-Saxon potential was compared with equation (10), we obtained:

$$\begin{aligned}D_0&=1-\left(\frac{1+e^{-aR_0}}{aR_0}\right)^2\left(\frac{4aR_0}{1+e^{-aR_0}}-3-aR_0\right)\\D_1&=2\left(1+e^{aR_0}\right)\left[\frac{3\left(1+e^{-aR_0}\right)}{aR_0}-\left(3+aR_0\right)\frac{\left(1+e^{-aR_0}\right)}{aR_0}\right]\\D_2&=\left(1+e^{aR_0}\right)^2\left(\frac{1+e^{-aR_0}}{aR_0}\right)^2\left(3+aR_0-\frac{2aR_0}{1+e^{-aR_0}}\right)\end{aligned} \tag{11}$$

Introducing the new variable $y=\frac{1}{1+e^{x/a}}$, the Woods-Saxon potential was expressed as:

$$V_{eff}(r)=-v_0y+\delta^2\left(D_0-D_1y+D_2y^2\right) \tag{12}$$

Where $\delta^2=\left(l+\frac{D-1}{2}\right)\left(l+\frac{D-3}{2}\right)\Big/R_0^2$.

We can obtain two turning points by solving equation $E_{n,l}^D=V(r)$:

$$y_A + y_B = \frac{\nu_0 + \delta^2 D_1}{\delta^2 D_2}, \qquad y_A \cdot y_B = \frac{\delta^2 D_0 - E_{n,l}^D}{\delta^2 D_2} \tag{13}$$

Substituting the new variable into the ground state of the nonlinear Riccati equation (2), we can obtain:

$$\frac{y^2 - y}{a}\frac{d\phi_0(r)}{dy} = -\left[E_0 + \nu_0 y - \delta^2\left(D_0 - D_1 y + D_2 y^2\right)\right] - \phi_0(r)^2 \tag{14}$$

Since the ground state $\phi_0(r)$ has only one zero point and no pole, it had to take the linear form in *y*. According to the Sturm-Liouville theorem, after taking $\phi_0(r) = C_1 y + C_2$ ($C_1 > 0$), substituting this into equation (14), we can obtain:

$$E_0 = -C_2^2 + \delta^2 D_0 \tag{15}$$

$$C_1 = \frac{\sqrt{1 + 4\delta^2 a^2 D_2} - 1}{2a} \qquad C_2 = \frac{1}{2a} - \frac{\delta^2 D_1 + \nu_0}{2C_1} \quad (C_1 > 0) \tag{16}$$

The wave vector *k*(*r*) can be expressed as:

$$k(r) = \delta\sqrt{D_2}\sqrt{(y_B - y)(y - y_A)} \tag{17}$$

Substituting it into the first integral of equation (7), we obtained:

$$\begin{aligned}\int_{r_A}^{r_B} k(r)\,dr &= \int_{r_A}^{r_B} \sqrt{E_{n,l}^D - V_{eff}(r)}\,dr \\ &= a\delta\sqrt{D_2}\int_{y_A}^{y_B} \frac{\sqrt{(y_B - y)(y - y_A)}}{y(y-1)}\,dy \\ &= a\delta\pi\sqrt{D_2}\left[\sqrt{y_A y_B} + \sqrt{(1 - y_A)(1 - y_B)} - 1\right]\end{aligned} \tag{18}$$

The integral formula $\int_{z_A}^{z_B} \frac{\sqrt{(z - z_A)(z_B - z)}}{z(1-z)}\,dz = \pi\left[-\sqrt{z_A z_B} + 1 - \sqrt{(1 - z_A)(1 - z_B)}\right]$ can be used calculating.

Substituting equation (13) into equation (18), the equation (19) can be obtained:

$$\int_{r_A}^{r_B} k(r)\,dr = a\pi\left[\sqrt{\delta^2\left(D_0 - D_1 + D_2\right) - E_{n,l}^D - \nu_0} + \sqrt{\delta^2 D_0 - E_{n,l}^D} - \delta\right] \tag{19}$$

By replacing $E_{n,l}^D$ in equation (19) with E_0, we had:

$$\int_{r_{0A}}^{r_{0B}} k_0(r)\,dr = a\pi\left[\sqrt{\delta^2\left(D_0 - D_1 + D_2\right) - E_0 - \nu_0} + \sqrt{\delta^2 D_0 - E_0} - \delta\right] \tag{20}$$

Substituting equations (19) and (20) into equation (7) according to the proper quantization rule, we obtain:

$$\begin{aligned}&\int_{r_A}^{r_B} k(r)\,dr - \int_{r_{0A}}^{r_{0B}} k_0(r)\,dr \\ &= a\pi\left[\begin{array}{l}\sqrt{\delta^2\left(D_0 - D_1 + D_2\right) - E_{n,l}^D - \nu_0} + \sqrt{\delta^2 D_0 - E_{n,l}^D} \\ -\sqrt{\delta^2\left(D_0 - D_1 + D_2\right) - E_0 - \nu_0} - \sqrt{\delta^2 D_0 - E_0}\end{array}\right] \\ &= n\pi\end{aligned} \tag{21}$$

The solution can be obtained:

$$\begin{aligned}E_{n,l}^D &= \delta^2\left(D_0 - D_1 + D_2\right) - \nu_0 \\ &- \left[\frac{\left(\frac{n}{a} + \sqrt{\delta^2 D_0 - E_0} + \sqrt{\delta^2\left(D_0 - D_1 + D_2\right) - E_0 - \nu_0}\right)^2 + \delta^2\left(D_2 - D_1\right) - \nu_0}{2\left(\frac{n}{a} + \sqrt{\delta^2 D_0 - E_0} + \sqrt{\delta^2\left(D_0 - D_1 + D_2\right) - E_0 - \nu_0}\right)}\right]^2\end{aligned} \tag{22}$$

If $2M$ and $\hbar^2$ were considered in the calculation process, the energy eigenvalue $E_{n,l}^D$ in any dimension space can be expressed as:

$$E_{n,l}^D = \frac{\hbar^2\delta^2}{2M}\left(D_0 - D_1 + D_2\right) - \nu_0 - \frac{\hbar^2}{8Ma^2}\left[\frac{a^2\left(\delta^2 D_2 - \delta^2 D_1 + \frac{2M\nu_0}{\hbar^2}\right) - \Lambda^2}{\Lambda}\right]^2 \tag{23}$$

Where:

$$\begin{aligned}\Lambda &= n + aC_2 + a\sqrt{C_2^2 - \delta^2\left(D_1 - D_2\right) + \frac{2M}{\hbar^2}\nu_0} \\ C_1 &= \sqrt{\frac{1}{4a^2} + \frac{\hbar^2}{2M}\delta^2 D_2} - \frac{1}{2a} \\ C_2 &= \frac{1}{2a} - \frac{\frac{\hbar^2}{2M}\delta^2 D_1 + \nu_0}{2C_1} \\ \delta^2 &= \left(l + \frac{D-1}{2}\right)\left(l + \frac{D-3}{2}\right)\Big/ R_0^2\end{aligned} \tag{24}$$

CONCLUSION

Without solving complex differential equations, the energy spectrum of the *l*-state Woods-Saxon potential Schrödinger equation in *D*-dimensional space was calculated by using the proper quantization rule. This provided a possibility to study the complete energy spectrum of the cross-latitude degradation of the Woods-Saxon potential. It can be found from equations (23) and (24) that after replacing l, $D \Leftrightarrows l+i$, D-2i (i=0, 1, 2, 3...), the energy expression remains unchanged. This means that the angular momentum l of the Woods-Saxon potential changed with the dimension D, and the angular momentum was increased by one unit equal to the dimension by two units. For the Woods-Saxon potential, the parameters D and l satisfy $k \equiv D+2l$, which means that the energy eigenvalue depends on k and n. Therefore, if k was chosen as the

expansion parameter for the power function potential, the energy eigenvalue can be expanded to the expression of the variable *k* when *n* was large.

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